Separation Axioms and ij - α - Open Sets in Bitopological Spaces

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Abstract :

In this paper, we study especial cases of separation axioms in bitopological spaces by considering ij- α - open sets , we prove some results about them comparing with similar cases in topological spaces .

1.Introduction :

The study of bitopological spaces was initiated by Kelly , J . C . [3] . A triple (G,τ_1,τ_2) is called bitopological space if (G,τ_1) and (G,τ_2) are two topological spaces. α - open (resp . β - open) set in topological space defined by Njastad , O . (1965) [6] is as follows : A subset A of a topological space (G,τ) is said to be α - open (resp . β - open) set if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ (resp . $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$) , and the family of all α - open (resp . β - open) sets of G is denoted by $\alpha.O(G)$ (resp . $\beta.O(G)$) .

ij - α - open set in bitopological space defined by Jelic , M . (1990) [2] ; Kumar Sampath , S . (1997) [4] ; Nasef , A . A .and Noiri , T . (1998) [5] is as follows : A subset A of a bitopological space (G,τ_1,τ_2) is called ij - α - open set if $A \subseteq \text{i - int}(\text{j - cl}(\text{i - int}(A)))$, and the family of all ij - α - open sets of G is denoted by ij - $\alpha.O(G)$, where $\text{i} \neq \text{j}$; $\text{i , j} = 1,2$.

In this paper we first (in section 2) introduce a comparable studying between ij - α - open sets ,and then (in section 3) we study especial case of separation axioms in bitopological spaces in the sense of ij - α - open sets .

2. ij - α - Open Sets :

This section deals with the main definitions ij - α - open and ij- α - closed . It also presents a general study of the notions ij -α - neighborhood and ij - α - closure as well as some theorems and properties that are included throughout the work .

Definition 2.1 **:** [2]

A subset A of a bitopological space (G,τ_1,τ_2) is called ij - α - open set if $A \subseteq \text{i-int}(\text{j-cl}(\text{i-int}(A)))$, and the family of all ij - α - open sets of G is denoted by ij - $\alpha.O(G)$, where $\text{i} \neq \text{j}$; $\text{i},\text{j} = 1, 2$.

Example 2.2 **:**

Let $G = \{a,b,c\}$, $\tau_1 = \{G,\phi,\{a\}\}$,and $\tau_2 = \{G,\phi,\{a\},\{a,b\}\}$.

$(G,\tau_1)$and (G,τ_2) are two topological spaces , then (G,τ_1,τ_2) is a bitopological space .The family of all 12 - α - open sets and 21 - α - open sets of G are :

$12\text{-}\alpha.O(G) = 21\text{-}\alpha.O(G) = \{G,\phi,\{a\},\{a,b\},\{a,c\}\}$.

If we put $A = \{a\}$, then $\tau_1\text{-int}(\{a\}) = \{a\}, \tau_2\text{-cl}(\tau_1\text{-int}(\{a\})) = G$, so

$\tau_1\text{-int}(\tau_2\text{-cl}(\tau_1\text{-int}(\{a\}))) = \tau_1\text{-int}(G) = G$, there fore

$A \subseteq \tau_1\text{-int}(\tau_2\text{-cl}(\tau_1\text{-int}(A)))$.

$\tau_2\text{-int}(\{a\}) = \{a\}, \quad \tau_1\text{-cl}(\tau_2\text{-int}(\{a\})) = \tau_1\text{-cl}(\{a\}) = G$, so

$\tau_2\text{-int}(\tau_1\text{-cl}(\tau_2\text{-int}(\{a\}))) = \tau_2\text{-int}(G) = G$, there fore

$A \subseteq \tau_2\text{-int}(\tau_1\text{-cl}(\tau_2\text{-int}(A)))$.

Hence $A = \{a\}$ is 12 - α - open set and 21 - α - open set in (G,τ_1,τ_2).

In general in any bitopological space , G and ϕ are clearly ij - α - open sets for $\text{i},\text{j} \in \{1,2\}$.

Remark 2.3 **:**

In general $12\text{-}\alpha.O(G) \neq 21\text{-}\alpha.O(G)$ as in the following example .

Example 2.4 **:**

Let $G = \{a,b,c\}$, $\tau_1 = \{G,\phi,\{a\}\}$,and $\tau_2 = \{G,\phi,\{b\},\{c\},\{b,c\}\}$.

(G,τ_1) and (G,τ_2) are two topological spaces , then (G,τ_1,τ_2) is a bitopological space , it is easy to see that $\{a\}$ is a 12 - α - open set but not 21 - α - open set .

The following lemma will give an equivalent definition of ij - α - open sets .

***Lemma 2.5* :**

Asubset A of a bitopological space (G,τ_1,τ_2) is ij - α - open set iff there exist $B \in \tau_i$ such that $B \subseteq A \subseteq \text{i - int}(\text{j - cl}(B))$.

***Proof* :**

See [1] .

***Lemma 2.6* :**

$\text{i - int}(\text{j - cl}(\text{i - int}(\text{j - cl}(A)))) = \text{i - int}(\text{j - cl}(A))$.

***Proof* :**

See [1] .

***Lemma 2.7* :**

If B is ij- α - open set and $B \subseteq A \subseteq \text{i - int}(\text{j - cl}(B))$, then A is also ij - α - open set .

***Proof* :**

See [1] .

Lemma 2.8 :

Every τ_i - open set is ij - α - open set , but the converse is not true .

***Proof* :**

See [1] .

But the converse is not true as in the following example .

Example 2.9 :

In example (2.2) , $\{a,c\}$ is a 12 - α - open set but not τ_1 - open set .

Remark 2.10 :

The intersection of any two ij - α - open sets is not necessary ij - α - open set as in the following example .

Example 2.11 **:**

Let $G=\{a,b,c,d\}$, $\tau_1=\{G,\phi,\{a\},\{d\},\{a,d\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$, $\tau_2=\{G,\phi,\{a\},\{d\},\{a,d\}\}$.

(G,τ_1) , (G,τ_2) are two topological spaces , then (G,τ_1,τ_2) is a bitopological space . Hence $\{a,c\}$ and $\{b,c\}$ are two 12 - α - open sets ,

but $\{a,c\}\cap\{b,c\}=\{c\}$ is not 12 - α - open set .

Proposition 2.12 **:**

The union of any family of ij - α - open sets is ij - α - open set .

Proof **:**

See [1] .

Definition 2.13 **:** [4]

Let (G,τ_1,τ_2) be a bitopological space A subset A of G is called ij - α - closed set of G iff the complement of A is ij - α - open set of G , and the family of all ij - α - closed sets of G is denoted by ij - $\alpha.C(G)$.

Remark 2.14 **:**

The intersection of any family of ij - α - closed sets is ij - α - closed set . By proposition (2.12) .

Definition 2.15 **:**

Let (G,τ_1,τ_2) be a bitopological space , and let $g\in G$. A subset N of G is said to be ij - α - nhd of a point g iff there exists a ij - α - open set U such that $g\in U\subseteq N$. The set of all ij - α - nhds of a point g is denoted by ij - α - $N(g)$.

Lemma 2.16 **:**

Let (G,τ_1,τ_2) be a bitopological space and let $g\in G$. Then every τ_i - nhd of g is ij - α - nhd of g .

Proof **:**

See [1] .

But the converse is not true as in the following example .

Example 2.17 **:**

Let (G,τ_1,τ_2) be the same bitopological space of example (2.2) we have

$12\text{-}\alpha\text{-}N(a)=21\text{-}\alpha\text{-}N(a)=\{G,\{a\},\{a,b\},\{a,c\}\}$,

$12\text{-}\alpha\text{-}N(b)=21\text{-}\alpha\text{-}N(b)=\{G,\{a,b\}\}$,

$12\text{-}\alpha\text{-}N(c)=21\text{-}\alpha\text{-}N(c)=\{G,\{a,c\}\}$, and

$\tau_1\text{-}N(a)=\{G,\{a\},\{a,b\},\{a,c\}\}$,

$\tau_1\text{-}N(b)=\{G\}$,

$\tau_1\text{-}N(c)=\{G\}$;

$\tau_2\text{-}N(a)=\{G,\{a\},\{a,b\},\{a,c\}\}$,

$\tau_2\text{-}N(b)=\{G,\{a,b\}\}$,

$\tau_2\text{-}N(c)=\{G\}$.

Clearly $12\text{-}\alpha\text{-}N(c)$ is not equal to $\tau_1\text{-}N(c)$.

Proposition 2.18 **:**

$B\in \text{ij-}\alpha.O(G)$ iff B is an $\text{ij-}\alpha\text{-nhd}$ of each of its points .

Proof **:**

See [1] .

Definition 2.19 **:** [4]

Let (G,τ_1,τ_2) be a bitopological space and $A\subseteq G$, the intersection of all $\text{ij-}\alpha$-closed sets containing A is called $\text{ij-}\alpha$-closure of A , and is denoted by $\text{ij-}\alpha\text{-cl}(A)$; i.e $\text{ij-}\alpha\text{-cl}(A)=\bigcap\{B\subseteq G : B \text{ is ij-}\alpha\text{-closed set}, A\subseteq G\}$.

Example 2.20 **:**

By example (2.2) , the family of all $12\text{-}\alpha$-closed sets and $21\text{-}\alpha$-closed sets of G are : $12\text{-}\alpha.C(G)=21\text{-}\alpha.C(G)=\{G,\phi,\{b,c\},\{c\},\{b\}\}$.

If we take $A=\{b,c\}$, then $12\text{-}\alpha\text{-cl}(A)=21\text{-}\alpha\text{-cl}(A)=G\bigcap\{b,c\}=\{b,c\}$.

Theorem 2.21 **:**

Let (G,τ_1,τ_2) be a bitopological space , and let $A \subseteq G$, then :

(i) ij - α - cl(A) is the smallest ij - α - closed set containing A .

(ii) A is ij - α - closed set iff ij - α - cl$(A) = A$.

Proof **:**

See [1].

3.Separation Axioms in Bitopological Spaces :

In this section ij - α - T_0 , ij - α - T_1, ij - α - T_2 , ij -α - normal , ij - α - regular , ij - α - T_3 and ij - α - T_4 spaces are introduced , with several properties .

Definition 3.1 **:**

Let G be a bitopological space . Then G is called :

(i) ij - α - T_0 - space iff for each pair of distinct points in G , there exists an ij - α - open set in G containing one and not the other .

(ii) ij - α - T_1 - space iff for each pair of distinct points g and h , there exists an ij - α - open sets U and V containing g and h respectively such that $h \notin U$ and $g \notin V$.

(iii) ij - α - T_2 - space (ij - α - Hausdorff space) iff for each pair of distinct points g and h , there exist disjoint ij - α - open sets A and B in G such that $g \in A$ and $h \in B$.

(iv) ij - α - regular space iff for each ij - α - closed set A and for each $g \notin A$, there exist disjoint ij - α - open sets U and V such that $g \in U$ and $A \subseteq V$.

(v) ij - α - normal space iff for each pair of disjoint ij - α - closed sets A and B , there exist disjoint ij - α - open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(vi) ij - α - T_3 - space iff it is ij - α - T_1 and ij - α - regular .

(vii) ij - α - T_4 - space iff it is ij - α - T_1 and ij - α - normal .

***Theorem 3.2* :**

Every subspace of ij - α - T_k - space is ij - α - T_k - space , that is property is hereditary , where $k = 0,1,2$.

***Remark 3.3* :**

Every ij - α - T_1 - space is ij - α - T_0 - space , but the converse is not true as the following example .

***Example 3.4* :**

By example (2.2) , if we take a and b , $a \neq b$, then we can not find two 12 - α - open sets , such that one of them contains a but not b and the other contains b but not a .There fore (G,τ_1,τ_2) is not 12 - α - T_1 - space , but it is clear that (G,τ_1,τ_2) is 12 - α - T_0 - space .

***Proposition 3.5* :**

A bitopological space (G,τ_1,τ_2) is ij - α - T_1 - space iff a singleton subset $\{g\}$ of G is ij - α - closed .

***Proof* :**

See [1] .

***Remark 3.6* :**

Every ij - α - T_2 - space is ij - α - T_1 - space , but the converse is not true as the following example .

***Example 3.7* :**

Let $\tau_1 = \tau_2$ = the co-finite topology on an infinite set G . There fore (G,τ_1,τ_2) is not 12 - α - T_2 - space , but it is clear that (G,τ_1,τ_2) is 12 - α - T_1 - space .

***Corollary 3.8* :**

Each singleton subset of ij - α - T_2 - space is ij - α - closed .

Proof **:**

By proposition (3.5) and remark (3.6) .

Theorem 3.10 **:**

The property of a space being ij - α - regular space is hereditary .

Proof **:**

See [1] .

Theorem 3.11 **:**

Let (G,τ_1,τ_2) be a bitopological space , then (G,τ_1,τ_2) is ij - α - regular space iff for each ij - α - open set U and $g \in U$, there exists ij - α - open set V such that $g \in V$, ij - α - $\mathrm{cl}(V) \subseteq U$.

Proof **:**

See [1] .

Theorem 3.12 **:**

Every ij - α - closed subspace of ij - α - normal space is ij - α - normal space .

Proof **:**

See [1] .

Theorem 3.13 **:**

The property of a space being ij - α - T_3 - space is hereditary .

Proof **:**

See [1] .

Remark 3.14 **:**

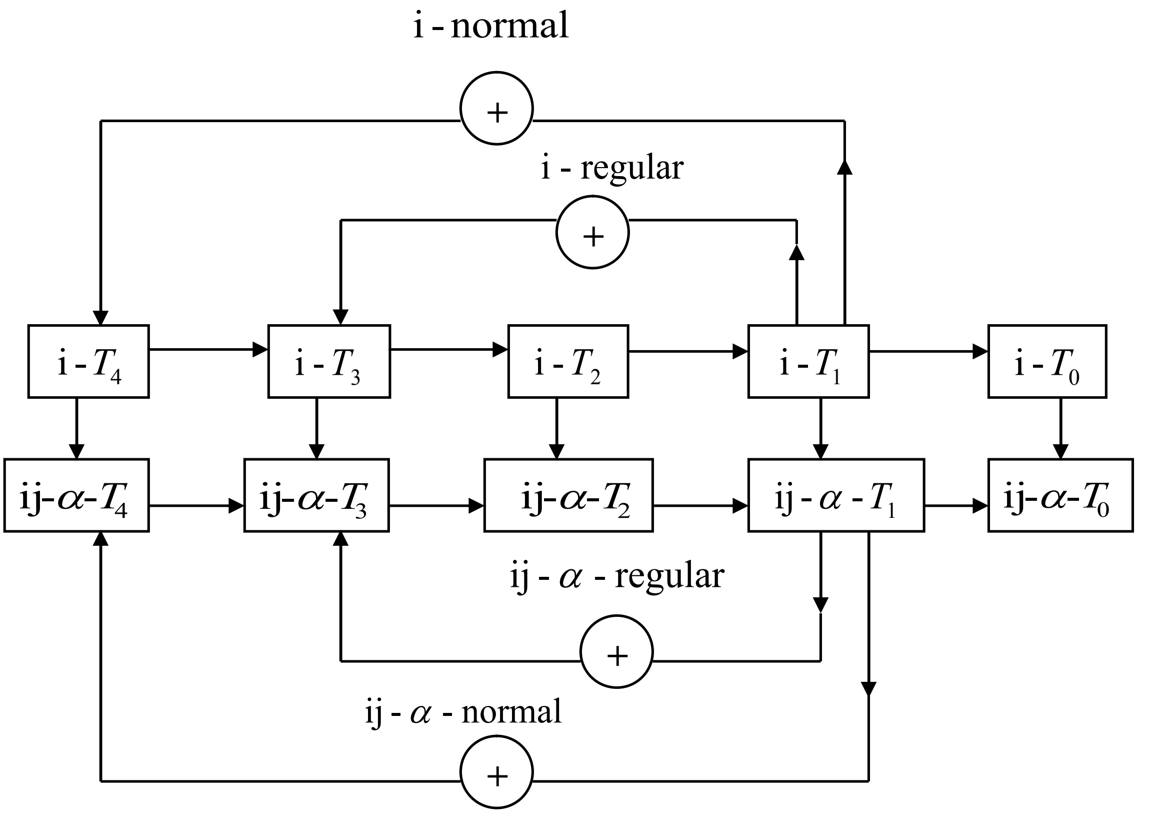
Every ij - α - T_3 - space is ij - α - T_2 - space , but the converse is not true .

Remark 3.15 **:**

Every ij - α - T_4 - space is ij - α - T_3 - space , but the converse is not true .

Remark 3.16 **:**

The following diagram shows the relations between $i\text{-}T_0$, $i\text{-}T_1$, $i\text{-}T_2$, $i\text{-}T_3$, $i\text{-}T_4$, $ij\text{-}\alpha\text{-}T_0$, $ij\text{-}\alpha\text{-}T_1$, $ij\text{-}\alpha\text{-}T_2$, $ij\text{-}\alpha\text{-}T_3$ and $ij\text{-}\alpha\text{-}T_4$ spaces :



And no other relations hold between them .

4. References :

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