Semi - α - Separation Axioms in Bitopological Spaces

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Abstract :

In this paper, we study especial cases of separation axioms in bitopological spaces by considering ij - semi - α - open sets , we prove some results about them comparing with similar cases in topological spaces .

Key words : Bitopological space , ij - semi - α - open set , ij - semi- α - T_k - space (k = 0,1,2 ,3,4) , ij - semi- α - regular , ij - semi- α - normal .

1. Introduction :

The study of bitopological spaces was initiated by Kelly, J .C ., [6] . A triple (X,τ_1,τ_2) is called bitopological space if (X,τ_1) and (X,τ_2) are two topological spaces . In 1991, G . B . Navalagi [2] introduced the concept of semi- α - open sets in topological spaces . In 1990 ,Jelic , M . [4] introduced the concept of ij - α - open sets in bitopological spaces . In 1981 , Bose , S. , [1] introduced the notion of ij - semi - open sets in bitopological spaces . In 1992 , Kar A ., [5] have introduced the notion of ij - pre - open sets in bitopological spaces .

In this paper , we introduce the concept of ij - semi - α - open sets in a bitopological spaces and their relationships with ij - semi - open sets , ij - pre - open sets and ij - α - open sets are studied we study especial cases of separation axioms in bitopological spaces by considering ij- semi - α - open sets .

2. Preliminaries :

Throughout the paper , spaces always mean a bitopological spaces , the closure and the interior of any subset A of X with respect to τ_i, will be denoted by $\tau_i - cl(A)$, and $\tau_i - \text{int}(A)$ respectively, for i = 1,2 .

Definition 2.1 : [2,9]

Let (X,τ) be a topological space, $A \subseteq X$. Then A is a said to be semi- α -open set if and only if there exists an α -open set U in X , such that $U \subseteq A \subseteq cl(U)$. The family of all semi- α -open sets of X is denoted by $S_\alpha O(X)$.

Definition 2.2 :

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$, A is said to be :

(i) ij - pre - open set [5] if $A \subseteq i-\text{int}(j-cl(A))$, where $i \neq j; i, j = 1,2$,

(ii) ij - semi - open set [1] if $A \subseteq j-cl(i-\text{int}(A))$, where $i \neq j; i, j = 1,2$,

(iii) ij - α - open set [4,7] if $A \subseteq i-\text{int}(j-cl(i-\text{int}(A)))$, where $i \neq j; i, j = 1,2$.

Remark 2.3 :

The family of ij - pre - open (resp. ij - semi - open and ij - α - open) sets of X is denoted by ij - $PO(X)$ (resp. ij - $SO(X)$ and ij - $\alpha O(X)$) , where $i \neq j; i, j = 1,2$.

Example 2.4 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\}\}$, and $\tau_2 = \{X,\phi,\{b\},\{c\},\{b,c\}\}$.

(X,τ_1) and (X,τ_2) are two topological spaces , then (X,τ_1,τ_2) is a bitopological space .

The family of all 12 - pre - open sets of X is : 12 - $PO(X) = \{X,\phi,\{a\},\{b,c\}\}$.

The family of all 12 - semi - open sets of X is : 12 - $SO(X) = \{X,\phi,\{a\}\}$.

The family of all 12 - α - open sets of X is : 12 - $\alpha O(X) = \{X,\phi,\{a\}\}$.

Definition 2.5 :

The complement of an ij - pre - open (resp. ij - semi - open and ij - α - open) set is said to be ij - pre - closed (resp. ij - semi - closed and ij - α - closed) set . The family of ij - pre - closed (resp. ij - semi - closed and ij - α - closed) sets of X is denoted by ij - $PC(X)$ (resp. ij - $SC(X)$ and ij - $\alpha C(X)$) , where $i \neq j; i, j = 1,2$.

Remark 2.6 :

It is clear by definition that in any bitopological space the following hold :

(i) every τ_i – open set is ij - pre - open , ij - semi - open , ij - α - open set .

(ii) every ij - α - open set is ij - pre - open , ij - semi - open set .

(iii) the concept of ij - pre - open and ij - semi - open sets are independent .

Proposition 2.7 :

A subset A of a bitopological space (X,τ_1,τ_2) is ij - α - open set if and only if there exists an τ_i – open set U , such that $U \subseteq A \subseteq i-\text{int}(j-cl(U))$.

Proof :

This follows directly from the definition (2.2) (iii) . ■

Proposition 2.8 : [8]

A subset A of a bitopological space (X,τ_1,τ_2) is ij - semi - open set if and only if there exists an τ_i – open set U, such that $U \subseteq A \subseteq j-cl(U)$.

Proposition 2.9 : [3]

A subset A of a bitopological space (X,τ_1,τ_2) is ij - pre - open set if and only if there exists an τ_i – open set U, such that $A \subseteq U \subseteq j-cl(A)$.

Theorem 2.10 :

A subset A of a bitopological space (X,τ_1,τ_2) is an ij - α - open set if and only if A is ij - semi - open set and ij - pre - open set .

Proof :

Follows from definition (2.2) and remark (2.6) . ■

3. Semi -α - Open Sets in Bitopological Spaces :

In this section the notion of ij - semi - α - open sets is introduced in bitopological spaces and their relationships with ij - semi - open sets , ij - pre - open sets and ij - α - open sets are studied .

Definition 3.1 :

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$. Then A is a said to be ij - semi - α - open set if there exists an ij - α - open set U in X , such that $U \subseteq A \subseteq j-cl(U)$. The family of all ij - semi - α - open sets of X is denoted by $ij\text{-}S_\alpha O(X)$, where $i \neq j; i, j = 1,2$.

Example 3.2 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\},\{a,b\}\}$, and $\tau_2 = \{X,\phi,\{b\},\{c\},\{b,c\}\}$.
(X,τ_1) and (X,τ_2) are two topological spaces , then (X,τ_1,τ_2) is a bitopological space .
The family of all 12 - semi - α - open sets of X is : $12\text{-}S_\alpha O(X) = \{X,\phi,\{a\},\{a,b\}\}$.
The family of all 21 - semi - α - open sets of X is : $21\text{-}S_\alpha O(X) = \{X,\phi,\{b\},\{c\},\{b,c\}\}$.

Remark 3.3 :

In general $12\text{-}S_\alpha O(X) \neq 21\text{-}S_\alpha O(X)$ as in the following example .

Example 3.4 :

In example (3.2) , it is easy to see that $\{a\}$ is a 12 - semi - α - open set but not 21 - semi - α - open set .

The following proposition will give an equivalent definition of ij - semi - α - open sets .

Proposition 3.5 :

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$. Then A is an ij - semi - α - open set if and only if $A \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$.

Proof :

Necessity , suppose that A is ij - semi - α - open set , then there exists an ij - α - open set U , such that $U \subseteq A \subseteq j-cl(U)$. Since U is an ij - α - open set , then $U \subseteq i-\text{int}(j-cl(i-\text{int}(U)))$.
Implies , $j-cl(U) \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(U))))$.
Since $U \subseteq A$, then $j-cl(i-\text{int}(j-cl(i-\text{int}(U)))) \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$.
Therefore $j-cl(U) \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$. But $A \subseteq j-cl(U)$, which implies $A \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$.
Sufficiency , suppose $A \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$, to prove A is an ij - semi - α - open .
Let $V = i-\text{int}(A)$, we know that $i-\text{int}(A) \subseteq A$, we must show that $A \subseteq j-cl(i-\text{int}(A))$.
Since $i-\text{int}(j-cl(i-\text{int}(A))) \subseteq j-cl(i-\text{int}(A))$, then $j-cl(i-\text{int}(j-cl(i-\text{int}(A)))) \subseteq$ $j-cl(j-cl(i-\text{int}(A))) = j-cl(i-\text{int}(A))$. But , $A \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$ (by hypothesis) implies $A \subseteq j-cl(i-\text{int}(A))$. Therefore , there exists an $\tau_i -$ open set V , such that $V \subseteq A \subseteq j-cl(V)$. On the other hand V is an ij - α - open set (since V is $\tau_i -$ open) , then A is ij - semi - α - open set . ■

Proposition 3.6 :

The union of any family of ij - semi - α - open sets is ij - semi - α - open set .

Proof :

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of ij - semi - α - open subsets of X .

Then $A_\lambda \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A_\lambda))))$, for every $\lambda \in \Lambda$. Since $\bigcup_{\lambda\in\Lambda}\text{int}(A_\lambda) \subseteq \text{int}(\bigcup_{\lambda\in\Lambda}A_\lambda)$ and $\bigcup_{\lambda\in\Lambda}cl(A_\lambda) \subseteq cl(\bigcup_{\lambda\in\Lambda}A_\lambda)$ hold for any topology .

We have $\bigcup_{\lambda\in\Lambda}A_\lambda \subseteq \bigcup_{\lambda\in\Lambda} j-cl(i-\text{int}(j-cl(i-\text{int}(A_\lambda))))$

$$\subseteq j-cl(\bigcup_{\lambda\in\Lambda} i-\text{int}(j-cl(i-\text{int}(A_\lambda))))$$

$$\subseteq j-cl(i-\text{int}(\bigcup_{\lambda\in\Lambda} j-cl(i-\text{int}(A_\lambda))))$$

$$\subseteq j-cl(i-\text{int}(j-cl(\bigcup_{\lambda\in\Lambda} i-\text{int}(A_\lambda))))$$

$$\subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(\bigcup_{\lambda\in\Lambda} A_\lambda)))) .$$

Hence $\bigcup_{\lambda\in\Lambda} A_\lambda$ is ij - semi - α - open set . ■

Remark 3.7 :

The intersection of any two ij - semi - α - open sets is not necessary ij - semi - α - open set as in the following example .

Example 3.8 :

Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$, and $\tau_2=\{X,\phi,\{a\}\}$. The family of all

12 - semi - α - open sets of X is : 12 - $S_\alpha O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$.

Hence $\{a,c\}$ and $\{b,c\}$ are two 12 - semi - α - open sets , but $\{a,c\}\cap\{b,c\}=\{c\}$ is not

12 - semi - α - open set .

Remark 3.9 :

(i) Every τ_i – open set is ij - semi - α - open set ,but the converse need not be true .

(ii) If every τ_i – open set is τ_i – closed and every nowhere τ_i – dense set is τ_i – closed in any bitopological space , then every ij - semi - α - open set is an τ_i – open set .

Remark 3.10 :

(i) Every ij - α - open set is ij - semi - α - open set , but the converse is not true in general .

(ii) If every τ_i – open set is τ_i – closed set in any bitopological space , then every ij - semi - α - open set is an ij - α - open set .

Remark 3.11 :

The concepts of ij - semi - α - open and ij - pre - open sets are independent , as the following example .

Example 3.12 :

In example (2.4) , $\{b,c\}$ is a 12 - pre - open set but not 12 - semi - α - open set .

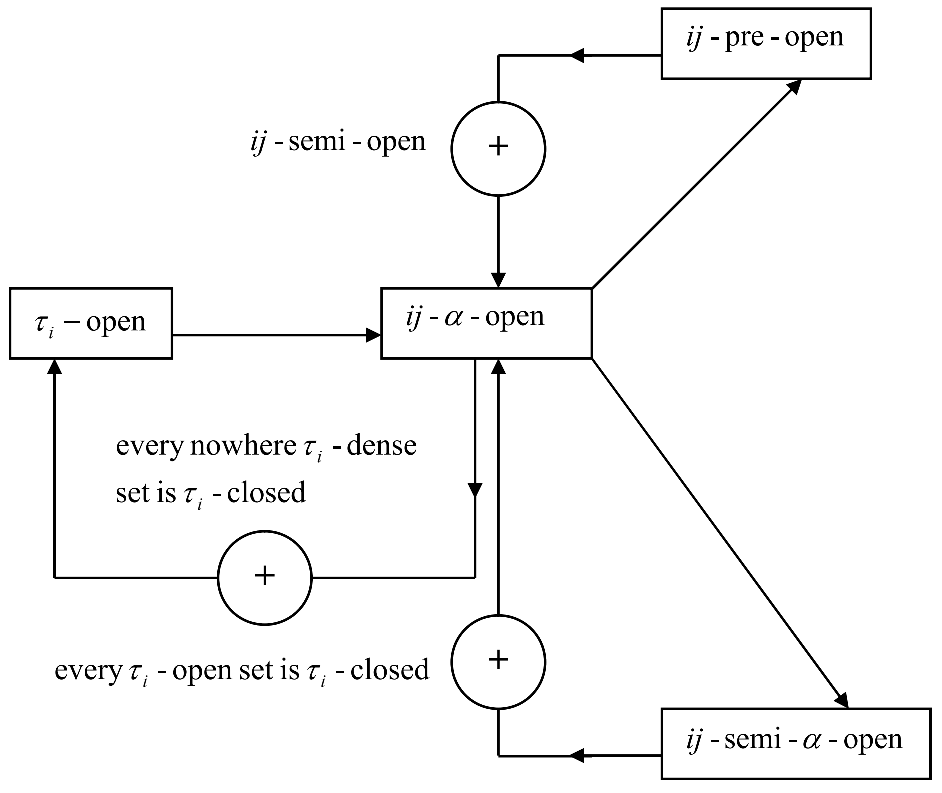
Remark 3.13 :

(i) It is clear that every ij - semi - open and ij - pre - open subsets of any bitopological space is ij - semi - α - open set (by theorem (2.10) and remark (3.10) (i)) .

(ii) An ij - semi - α - open set in any bitopological space (X,τ_1,τ_2) is ij - pre - open set if every τ_i – open subset of X is τ_i – closed set (from remark (3.10) (ii) and remark (2.6) (iii)) .

Remark 3.14 :

The following diagram shows the relations among the different types of weakly open sets that were studied in this section :



Definition 3.15 :

The complement of ij - semi - α - open set is called ij - semi - α - closed set . Then family of all ij - semi - α - closed sets of X is denoted by $ij\text{-}S_\alpha C(X)$, where $i \neq j; i, j = 1,2$.

Remark 3.16 :

The intersection of any family of ij - semi - α - closed sets is ij - semi - α - closed set .

Definition 3.17 :

Let (X,τ_1,τ_2) be a bitopological space and $A\subseteq X$,the intersection of all ij - semi - α - closed sets containing A is called ij - semi - α - closure of A , and is denoted by $ij\text{-}S_\alpha\text{-}cl(A)$;

i.e $ij\text{-}S_\alpha\text{-}cl(A)=\bigcap\{B\subseteq X : B \text{ is } ij\text{-semi-}\alpha\text{-closed set}, A\subseteq X\}$.

When confusion is possible as to what space the ij - semi - α - closure is be taken in , we shall write $ij\text{-}S_\alpha\text{-}cl_X(A)$.

Example 3.18 :

In example (3.8) , then family of all 12 - semi - α - closed sets of X is :

$12\text{-}S_\alpha C(X)=\{X,\phi,\{b,c\},\{a,c\},\{c\},\{b\},\{a\}\}$.

If we take $A=\{b,c\}$, then $12\text{-}S_\alpha\text{-}cl(A)=12\text{-}S_\alpha\text{-}cl(\{b,c\})=X\bigcap\{b,c\}=\{b,c\}$.

Theorem 3.19 :

Let (X,τ_1,τ_2) be a bitopological space , and let $A\subseteq X$,then :

(i) $ij\text{-}S_\alpha\text{-}cl(A)$ is the smallest ij - semi - α - closed set containing A .

(ii) A is ij - semi - α - closed set if and only if $ij\text{-}S_\alpha\text{-}cl(A)=A$.

Proof :

(i) This follows directly from the definition (3.17) .

(ii) If A is ij - semi - α - closed set , then A is it self is the smallest ij - semi - α - closed set containing A and hence $ij\text{-}S_\alpha\text{-}cl(A)=A$.

Conversely , if $ij\text{-}S_\alpha\text{-}cl(A)=A$. By (i) , $ij\text{-}S_\alpha\text{-}cl(A)$ is ij - semi - α - closed and so A is also ij - semi - α - closed set . ■

Remark 3.20 :

Let (X,τ_1,τ_2) be a bitopological space , and let A ,B be any subsets of X , then :

(i) $A\subseteq ij\text{-}S_\alpha\text{-}cl(A)$.

(ii) If $A\subseteq B$, then $ij\text{-}S_\alpha\text{-}cl(A)\subseteq ij\text{-}S_\alpha\text{-}cl(B)$.

(iii) $ij\text{-}S_\alpha\text{-}cl(ij\text{-}S_\alpha\text{-}cl(A))=ij\text{-}S_\alpha\text{-}cl(A)$.

Proof :

(i) By theorem (3.19) part (i) , we obtain $A\subseteq ij\text{-}S_\alpha\text{-}cl(A)$.

(ii) By a part (i) above , $B\subseteq ij\text{-}S_\alpha\text{-}cl(B)$, since $A\subseteq B$, then $A\subseteq ij\text{-}S_\alpha\text{-}cl(B)$ but $ij\text{-}S_\alpha\text{-}cl(B)$ is an ij - semi - α - closed set ,thus $ij\text{-}S_\alpha\text{-}cl(B)$ is ij - semi - α - closed set containing A .

Since $ij\text{-}S_\alpha\text{-}cl(A)$ is the smallest ij - semi - α - closed set containing A , hence $ij\text{-}S_\alpha\text{-}cl(A)\subseteq ij\text{-}S_\alpha\text{-}cl(B)$.

(iii) $ij\text{-}S_\alpha\text{-}cl(A)$ is ij-semi-α-closed set ,we have by theorem (3.19) part (ii) ,
$ij\text{-}S_\alpha\text{-}cl(ij\text{-}S_\alpha\text{-}cl(A)) = ij\text{-}S_\alpha\text{-}cl(A)$. ■

Definition 3.21 :

Let (X, τ_1, τ_2) be a bitopological space , and let Y be a subset of X.
The relative bitopological space for Y is denoted by $(Y, \tau_{1Y}, \tau_{2Y})$, such that :
$\tau_{1Y} = \{A \cap Y : A \in \tau_1\}$, and $\tau_{2Y} = \{B \cap Y : B \in \tau_2\}$ are two topologies for Y, called the relative topologies for Y . Then $(Y, \tau_{1Y}, \tau_{2Y})$ is called a subspace of bitopological space (X, τ_1, τ_2) , the relative bitopological space for Y with respect to ij-semi-α-open sets is the collection $ij\text{-}S_\alpha O(X)_Y$ given by : $ij\text{-}S_\alpha O(X)_Y = \{A \cap Y : A \in ij\text{-}S_\alpha O(X)\}$.

Remark 3.22 :

In general $ij\text{-}S_\alpha O(X)_Y \neq ij\text{-}S_\alpha O(Y)$, where $i \neq j; i, j = 1,2$.

Remark 3.23 :

Let $(Y, \tau_{1Y}, \tau_{2Y})$ be a subspace of a bitopological space (X, τ_1, τ_2) . Then :
(i) a subset A of Y is ij-semi-α-closed in Y iff there exists ij-semi-α-closed F in X such that $A = F \cap Y$.
(ii) for every $A \subseteq Y$, $ij\text{-}S_\alpha\text{-}cl_Y(A) = ij\text{-}S_\alpha\text{-}cl_X(A) \cap Y$.

Proof :

(i) A is ij-semi-α-closed in Y iff $Y - A = B \cap Y$ for some ij-semi-α-open subset B of X iff $A = Y - (B \cap Y) = (Y - B) \cup (Y - Y)$ [De – Morgan law] iff $A = Y - B$ iff $A = Y \cap (X - B)$ iff $A = Y \cap F$ (where $F = X - B$ is ij-semi-α-closed in X, since B is ij-semi-α-open in X) .
(ii) By definition (3.17) , $ij\text{-}S_\alpha\text{-}cl_Y(A) = \cap\{K : K \text{ is } ij\text{-semi-}\alpha\text{-closed in } Y \text{ and } A \subseteq K\}$
$= \cap\{F \cap Y : F \text{ is } ij\text{-semi-}\alpha\text{-closed in } X \text{ and } A \subseteq F \cap Y\}$ [by (i) above]
$= \cap\{F \cap Y : F \text{ is } ij\text{-semi-}\alpha\text{-closed in } X \text{ and } A \subseteq F\}$
$= [\cap\{F : F \text{ is } ij\text{-semi-}\alpha\text{-closed in } X \text{ and } A \subseteq F\}] \cap Y = ij\text{-}S_\alpha\text{-}cl_X(A) \cap Y$. ■

Remark 3.24 :

Let $(Y, \tau_{1Y}, \tau_{2Y})$ be a subspace of a bitopological space (X, τ_1, τ_2) .
If a subset A of Y is ij-semi-α-open (ij-semi-α-closed) in X, then A also ij-semi-α-open (ij-semi-α-closed) in Y .

Proof :

Since $A \subseteq Y$, we have $A = A \cap Y$ so that A is the intersection of Y with a set ij-semi-α-open (ij-semi-α-closed) in X, namely A . Hence by the definition (3.21) and remark (3.23) (i), A is ij-semi-α-open (ij-semi-α-closed) in Y . ■

4. Separation Axioms in Bitopological Spaces :

In this section ij - semi - α - T_0 , ij - semi - α - T_1, ij - semi - α - T_2, ij - semi - α - regular , ij - semi - α - normal , ij - semi - α - T_3 and ij - semi - α - T_4 spaces are introduced , with several properties .

Definition 4.1 :

Let (X,τ_1,τ_2) be a bitopological space . Then X is called :

(i) ij - semi - α - T_0 - space iff for each pair of distinct points in X, there exists an ij - semi - α - open set in X containing one and not the other .

(ii) ij - semi - α - T_1 - space iff for each pair of distinct points x and y in X, there exists an ij - semi - α - open sets G and H containing x and y respectively such that $y \notin G$ and $x \notin H$.

(iii) ij - semi - α - T_2 - space (ij - semi - α - Hausdorff space) iff for each pair of distinct points x and y , there exist disjoint ij - semi - α - open sets G and H in X such that $x \in G$ and $y \in H$.

(iv) ij - semi - α - regular space iff for each ij - semi - α - closed set A and for each $x \notin A$, there exist disjoint ij - semi - α - open sets G and H such that $x \in G$ and $A \subseteq H$.

(v) ij - semi - α - normal space iff for each pair of disjoint ij - semi - α - closed sets A and B , there exist disjoint ij - semi - α - open sets G and H such that $A \subseteq G$ and $B \subseteq H$.

(vi) ij - semi - α - T_3 - space iff it is ij - semi - α - T_1 and ij - semi - α - regular .

(vii) ij - semi - α - T_4 - space iff it is ij - semi - α - T_1 and ij - semi - α - normal .

Theorem 4.2 :

A bitopological space (X,τ_1,τ_2) is ij - semi - α - T_0 - space iff for each distinct points x , y in X, $ij\text{-}S_\alpha\text{-}cl(\{x\}) \neq ij\text{-}S_\alpha\text{-}cl(\{y\})$.

Proof :

Let $x, y \in X$, such that $x \neq y$ and let $ij\text{-}S_\alpha\text{-}cl(\{x\}) \neq ij\text{-}S_\alpha\text{-}cl(\{y\})$.
Then there exists at least one point z in X, such that $z \in ij\text{-}S_\alpha\text{-}cl(\{x\})$ but $z \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$. Suppose $z \in ij\text{-}S_\alpha\text{-}cl(\{x\})$. To show that $x \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$. If $x \in ij\text{-}S_\alpha\text{-}cl(\{y\})$, then $\{x\} \subseteq ij\text{-}S_\alpha\text{-}cl(\{y\})$, so $ij\text{-}S_\alpha\text{-}cl(\{x\}) \subseteq ij\text{-}S_\alpha\text{-}cl(ij\text{-}S_\alpha\text{-}cl(\{y\})) = ij\text{-}S_\alpha\text{-}cl(\{y\})$, hence $z \in ij\text{-}S_\alpha\text{-}cl(\{x\})$, then $z \in ij\text{-}S_\alpha\text{-}cl(\{y\})$ which is contradiction . Hence $x \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$, consequently $x \in X-(ij\text{-}S_\alpha\text{-}cl(\{y\}))$ but $ij\text{-}S_\alpha\text{-}cl(\{y\})$ is ij - semi - α - closed, so $X-(ij\text{-}S_\alpha\text{-}cl(\{y\}))$ is ij - semi - α - open which containing x but not y . It follows that (X,τ_1,τ_2) is ij - semi - α - T_0 - space .

Conversely, since (X,τ_1,τ_2) is ij - semi - α - T_0 - space , then for each two distinct points $x, y \in X$, there exists ij - semi - α - open set G such that $x \in G$, $y \notin G$. $X-G$ is ij - semi - α - closed set which does not contain x but contains y . By definition (3.17) , $ij\text{-}S_\alpha\text{-}cl(\{y\})$ is the intersection of all ij - semi - α - closed sets which containing $\{y\}$. Thus , $ij\text{-}S_\alpha\text{-}cl(\{y\}) \subseteq X-G$, then $x \notin X - G$. This implies that $x \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$, so we have $x \in ij\text{-}S_\alpha\text{-}cl(\{x\})$, $x \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$. There fore $ij\text{-}S_\alpha\text{-}cl(\{x\}) \neq ij\text{-}S_\alpha\text{-}cl(\{y\})$. ■

Theorem 4.3 :

Every subspace of ij - semi - α - T_0 - space is ij - semi - α - T_0 - space .

Proof :

Let $(Y, \tau_{1Y}, \tau_{2Y})$ be a subspace of a bitopological space (X, τ_1, τ_2) and let X is ij - semi - α - T_0 - space .To prove that Y is ij - semi - α - T_0 - space , let $y_1 \neq y_2 \in Y$. Since $Y \subseteq X$, then $y_1 \neq y_2 \in X$ and X is ij - semi - α - T_0 - space . There exists ij - semi - α - open set G in X, such that $y_1 \in G$ and $y_2 \notin G$. So $G \cap Y$ is ij - semi - α - open set in Y and $y_1 \in G \cap Y$, $y_2 \notin G \cap Y$. Hence $(Y, \tau_{1Y}, \tau_{2Y})$ is ij - semi - α - T_0 - space . ■

Remark 4.4 :

Every ij - semi - α - T_1 - space is ij - semi - α - T_0 - space , but the converse is not true .

Proof :

This follows directly from the definition (4.1) (i) , (ii) . ■

But the converse is not true as the following example .

Example 4.5 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, and $\tau_2 = \{X, \phi, \{a\}, \{a,b\}\}$, the family of all 12 - semi - α - open sets of X is : $12\text{-}S_\alpha O(X) = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ .If we take a and b , $a \neq b$, then we can not find two 12 - semi - α - open sets , such that one of them contains a but not b and the other contains b but not a . Therefore (X, τ_1, τ_2) is not 12 - semi - α - T_1 - space , but it is clear that (X, τ_1, τ_2) is 12 - semi - α - T_0 - space .

Theorem 4.6 :

Every subspace of ij - semi - α - T_1 - space is ij - semi - α - T_1 - space .

Proof :

Let $(Y, \tau_{1Y}, \tau_{2Y})$ be a subspace of a bitopological space (X, τ_1, τ_2) and let (X, τ_1, τ_2) is ij - semi - α - T_1 - space .To prove that Y is ij - semi - α - T_1 - space , let $y_1 \neq y_2 \in Y$. Since $Y \subseteq X$, then $y_1 \neq y_2 \in X$ and since X is ij - semi - α - T_1 - space . Then there exist two ij - semi - α - open sets G , H in X , such that $y_1 \in G$,but $y_2 \notin G$; and $y_2 \in H$,but $y_1 \notin H$ .Then we obtain two sets $G_1 = G \cap Y$, $H_1 = H \cap Y$ are ij - semi - α - open sets in Y , we have $y_1 \in G_1$, but $y_2 \notin G_1$; and $y_2 \in H_1$,but $y_1 \notin H_1$. Hence $(Y, \tau_{1Y}, \tau_{2Y})$ is ij - semi - α - T_1 - space . ■

Proposition 4.7 :

A bitopological space (X,τ_1,τ_2) is ij-semi-α-T_1- space iff every singleton subset $\{x\}$ of X is ij-semi-α-closed .

Proof :

Suppose X is ij-semi-α-T_1- space , and x be any point of X . Let $y \in X-\{x\}$, then $x \neq y$ and so there exists ij-semi-α-open set U containing y but not containing x ,and ij-semi-α-open set V containing x but not containing y , $y \in U \subseteq X-\{x\}$. Hence $X-\{x\}$ is ij-semi-α-open set , then $\{x\}$ is ij-semi-α-closed set .

Conversely , let $x, y \in X$, such that $x \neq y$. Since $\{x\}$ is ij-semi-α-closed set , then $X-\{x\}$ is ij-semi-α-open set containing y but not containing x . Similarly , $X-\{y\}$ is ij-semi-α-open set containing x but not containing y . Hence (X,τ_1,τ_2) is ij-semi-α-T_1- space . ■

Theorem 4.8 :

A bitopological space (X,τ_1,τ_2) is ij-semi-α-T_1- space iff $ij\text{-}S_\alpha\text{-}cl(\{a\})=\phi$, for each $a \in X$.

Proof :

Let (X,τ_1,τ_2) be ij-semi-α-T_1- space . Suppose $ij\text{-}S_\alpha\text{-}cl(\{a\})\neq\phi$ for some $a \in X$, then there exists a point b , such that $b \in ij\text{-}S_\alpha\text{-}cl(\{a\})$, $a \neq b$. Since X is ij-semi-α-T_1- space , then there exists ij-semi-α-open set G such that $a \notin G$, $b \in G$, thus $G \cap \{a\} = \phi$.

Hence , $b \notin ij\text{-}S_\alpha\text{-}cl(\{a\})$, which is contradiction . Thus $ij\text{-}S_\alpha\text{-}cl(\{a\})=\phi$.

Conversely , suppose that $ij\text{-}S_\alpha\text{-}cl(\{a\})=\phi$, for each $a \in X$, and let $x, y \in X$, such that $x \neq y$. Then $x \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$, and there exists ij-semi-α-open set G such that $x \in G$, $G \cap \{y\} = \phi$. Hence G contains x but not y .Similarly, there exists ij-semi-α-open set H contains y but not x . Thus (X,τ_1,τ_2) is ij-semi-α-T_1- space . ■

Remark 4.9 :

Every ij-semi-α-T_2- space is ij-semi-α-T_1- space , but the converse is not true .

Proof :

This follows directly from the definition (4.1) (ii) , (iii) . ■

But the converse is not true as the following example .

Example 4.10 :

Let X be an infinite set and $\tau_1 = \tau_2$ be the co-finite topology on X . Then (X,τ_1,τ_2) is 12-semi-α-T_1- space , but it is 12-semi-α-T_2- space .

Theorem 4.11 :

Every subspace of ij-semi-α-T_2- space is ij-semi-α-T_2- space .

Proof :

Let (X,τ_1,τ_2) be ij - semi - α - T_2 - space , and let $Y \neq \phi$ be a subset of X , and $x \neq y \in Y$, then $x, y \in X$, since (X,τ_1,τ_2) is ij - semi - α - T_2 - space , there exist two ij - semi - α - open sets G , H such that $x \in G$ and $y \in H$, $G \cap H = \phi$. So $G \cap Y$, $H \cap Y$ are ij - semi - α - open sets in Y and $x \in G \cap Y$, $y \in H \cap Y$; and $(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \phi$.
Hence (Y,τ_{1Y},τ_{2Y}) is ij - semi - α - T_2 - space . ■

Proposition 4.12 :

Each singleton subset of ij - semi - α - T_2 - space is ij - semi - α - closed .

Proof :

By proposition (4.7) and remark (4.9) . ■

Theorem 4.13 :

The property of a space being ij - semi - α - regular space is hereditary .

Proof :

Let (X,τ_1,τ_2) be ij - semi - α - regular space , let (Y,τ_{1Y},τ_{2Y}) be a subspace of X . To prove (Y,τ_{1Y},τ_{2Y}) is ij - semi - α - regular space ,let $y \in Y$, U be ij - semi - α - closed set in Y , such that $y \notin U$,then $ij\text{-}S_\alpha\text{-}cl_Y(U) = ij\text{-}S_\alpha\text{-}cl_X(U) \cap Y$, and since U is ij - semi - α - closed set in Y .
So $ij\text{-}S_\alpha\text{-}cl_Y(U) = U$. Then $U = ij\text{-}S_\alpha\text{-}cl_X(U) \cap Y$, since $y \notin U$, then $y \notin ij\text{-}S_\alpha\text{-}cl_X(U) \cap Y$, $y \notin ij\text{-}S_\alpha\text{-}cl_X(U)$, thus $ij\text{-}S_\alpha\text{-}cl_X(U)$ is ij - semi - α - closed set in X ; and since (X,τ_1,τ_2) is ij - semi - α - regular space , then there exist two disjoint ij - semi - α - open sets G , H in X , such that $y \in G$, $ij\text{-}S_\alpha\text{-}cl_X(U) \subseteq H$, $y \in G \cap Y$ and $ij\text{-}S_\alpha\text{-}cl_X(U) \cap Y \subseteq H \cap Y$, since G , H are ij - semi - α - open sets in X , then $G \cap Y$, $H \cap Y$ are ij - semi - α - open sets in Y .
Since $G \cap H = \phi$, then $(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \phi \cap Y = \phi$.
Hence (Y,τ_{1Y},τ_{2Y}) is ij - semi - α - regular space . ■

Proposition 4.14 :

Let (X,τ_1,τ_2) be a bitopological space , then (X,τ_1,τ_2) is ij - semi - α - regular space iff for each ij - semi - α - open set U and $x \in U$, there exists ij - semi - α - open set V such that $x \in V$, $ij\text{-}S_\alpha\text{-}cl(V) \subseteq U$.

Proof :

Let (X,τ_1,τ_2) be ij - semi - α - regular space , let $x \in U$ where is ij - semi - α - open set .
Let $A = X - U$, then A is ij - semi - α - closed , $x \notin A$ .Then there exists ij - semi - α - open sets W and V such that : $x \in V$, $A \subseteq W$, $V \cap W = \phi$.
Then $V \subseteq X - W$, $ij\text{-}S_\alpha\text{-}cl(V) \subseteq ij\text{-}S_\alpha\text{-}cl(X - W) = X - W$ ……………………….......(1)
$A \subseteq W$, then $X - W \subseteq X - A = U$, then $X - W \subseteq U$ ……………………………….(2)
From (1) and (2) we have : $x \in V$, $ij\text{-}S_\alpha\text{-}cl(V) \subseteq U$.

Conversely , let A be is ij - semi - α - closed and $x \notin A$. Let $U = X - A$, then U is ij - semi - α - open and $x \in U$. By hypothesis , there exists ij - semi - α - open V such that $x \in V$, $ij\text{-}S_\alpha\text{-}cl(V) \subseteq U$, $A \subseteq (X - (ij\text{-}S_\alpha\text{-}cl(V)))$, since $x \in V$, $V \cap (X - (ij\text{-}S_\alpha\text{-}cl(V))) = \phi$. Hence (X, τ_1, τ_2) is ij - semi - α - regular space . ■

Theorem 4.15 :

Every ij - semi - α - closed subspace of ij - semi - α - normal space is ij - semi - α - normal space .

Proof :

Let (X, τ_1, τ_2) be ij - semi - α - normal space and $(Y, \tau_{1Y}, \tau_{2Y})$ be any subspace of X . We have to show that $(Y, \tau_{1Y}, \tau_{2Y})$ is also ij - semi - α - normal space . Let L^*, M^* be disjoint ij - semi - α - closed subsets of Y . Then there exist ij - semi - α - closed subsets L , M of X , such that $L^* = L \cap Y$ and $M^* = M \cap Y$. Since Y is ij - semi - α - closed , it follows that L^*, M^* are disjoint ij - semi - α - closed subsets of X . Then by ij - semi - α - normality of X , there exists ij - semi - α - open sets A , B in X , such that $L^* \subseteq A, M^* \subseteq B$ and $A \cap B = \phi$. Since $L^* \subseteq Y$ and $M^* \subseteq Y$, these relations imply : $L^* \subseteq A \cap Y$, $M^* \subseteq B \cap Y$, $(A \cap Y) \cap (B \cap Y) = \phi$. Setting $A \cap Y = A^*$ and $B \cap Y = B^*$, we see that A^*, B^* are ij - semi - α - open sets of Y such that : $L^* \subseteq A^*$, $M^* \subseteq B^*$ and $A^* \cap B^* = \phi$. Accordingly $(Y, \tau_{1Y}, \tau_{2Y})$ is ij - semi - α - normal space . ■

Theorem 4.16 :

The property of a space being ij - semi - α - T_3 - space is hereditary .

Proof :

Let X be ij - semi - α - T_3 - space and Y be a subspace of X . Now X is ij - semi - α - regular space as well as ij - semi - α - T_1 - space and we have shown that both these properties are hereditary . It follows that Y is ij - semi - α - regular space as well as ij - semi - α - T_1 - space and hence it is ij - semi - α - T_3 - space . ■

Remark 4.17 :

Every ij - semi - α - T_3 - space is ij - semi - α - T_2 - space .

Proof :

This follows directly from the definition (4.1) (iii) , (vi) . ■

Remark 4.18 :

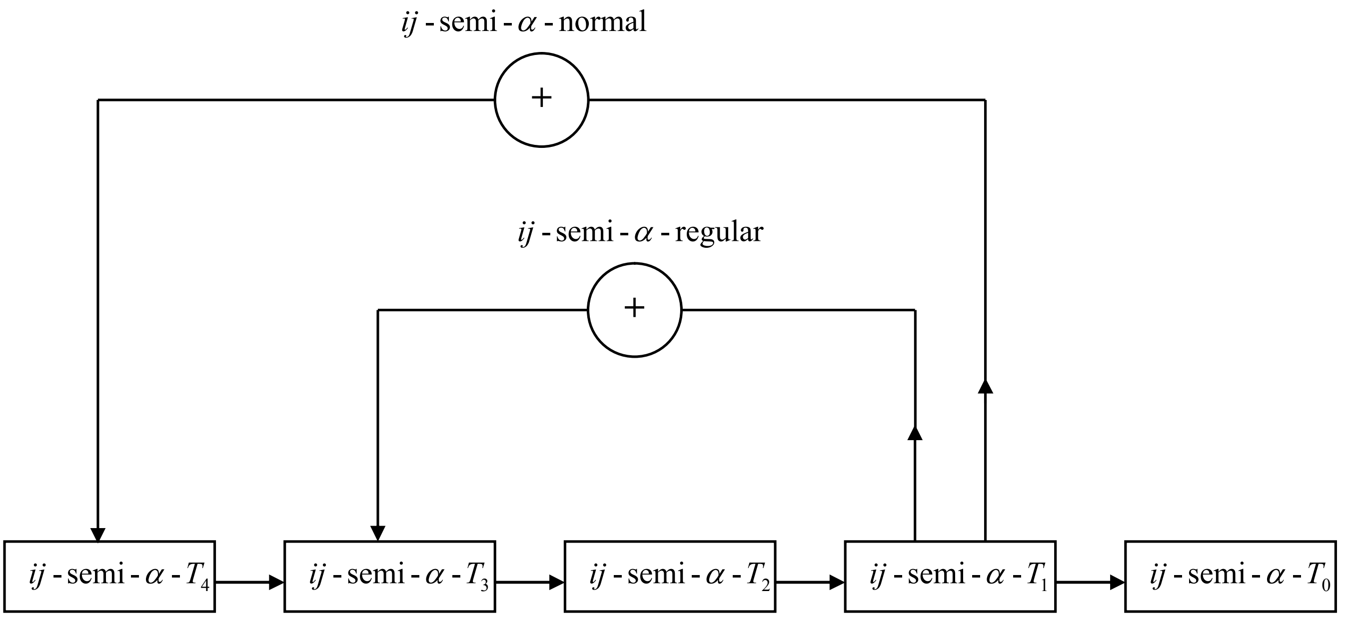
Every ij - semi - α - T_4 - space is ij - semi - α - T_3 - space .

Proof :

Let (X, τ_1, τ_2) be ij - semi - α - T_4 - space , then (X, τ_1, τ_2) is ij - semi - α - normal as well as ij - semi - α - T_1 - space . To prove that the space is ij - semi - α - T_3 - space , it suffices to show that the space is ij - semi - α - regular . Let F be ij - semi - α - closed subset of X and , let x be a point of X such that $x \notin F$. Since (X,τ_1,τ_2) is ij - semi - α - T_1 - space . Thus $\{x\}$ is ij - semi - α - closed subset of X , such that $\{x\} \cap F = \phi$,then by ij - semi - α - normality ,there exist ij - semi - α - open sets G , H such that $x \in G$, $F \subseteq H$ and $G \cap H = \phi$. It follows that the space (X,τ_1,τ_2) is ij - semi - α - regular space . ■

Remark 4.19 :

The following diagram shows the relations between ij - semi - α - T_0 , ij - semi - α - T_1, ij - semi - α - T_2 , ij - semi - α - T_3 and ij - semi - α - T_4 spaces :



And no other relations hold between them .

Definition 4.20 :

Let (X,τ_1,τ_2) be a bitopological space . Then X is called :

(i) ij - α - T_0 (resp . ij - pre - T_0) - space iff for each pair of distinct points in X , there exists an ij - α - open (resp. ij - pre - open) set in X containing one and not the other .

(ii) ij - α - T_1 (resp. ij - pre - T_1) - space iff for each pair of distinct points x and y , there exist an ij - α - open (resp. ij - pre - open) sets G and H containing x and y respectively such that $y \notin G$ and $x \notin H$.

(iii) ij - α - T_2 (resp. ij - pre - T_2) - space (ij - α - Hausdorff (resp. ij - pre - Hausdorff) space) iff for each pair of distinct points x and y , there exist disjoint ij - α - open (resp. ij - pre - open) sets G and H in X such that $x \in G$ and $y \in H$.
(iv) ij - α - regular (resp. ij - pre - regular) space iff for each ij - α - closed (resp. ij - pre - closed) set A and for each $x \notin A$, there exist disjoint ij - α - open (resp. ij - pre - open) sets G and H such that $x \in G$ and $A \subseteq H$.
(v) ij - α - normal (resp. ij - pre - normal) space iff for each pair of disjoint ij - α - closed (resp. ij - pre - closed) sets A and B , there exist disjoint ij - α - open (resp. ij - pre - open) sets G and H such that $A \subseteq G$ and $B \subseteq H$.
(vi) ij - α - T_3 (resp. ij - pre - T_3) - space iff it is ij - α - T_1 (resp. ij - pre - T_1) - space and ij - α - regular (resp. ij - pre - regular) space .
(vii) ij - α - T_4 (resp. ij - pre - T_4) - space iff it is ij - α - T_1 (resp. ij - pre - T_1) - space and ij - α - normal (resp. ij - pre - normal) space .

Remark 4.21 :

Every i - T_k - space is ij - pre - T_k - space , ij - α - T_k - space , ij - semi - α - T_k - space ,where $k = 0,1,2,3,4$.

Proof :

Follows from remark (2.6) (i) and remark (3.9) (i) . ■

Remark 4.22 :

Every ij - α - T_k - space is ij - semi - α - T_k - space , where $k = 0,1,2,3,4$.

Proof :

Follows from remark (3.10) (i) . ■

Remark 4.23 :

If every τ_i – open set is τ_i – closed set in any bitopological space , then every ij - semi - α - T_k - space is ij - α - T_k - space , where $k = 0,1,2,3,4$.

Proof :

Follows from remark (3.10) (ii) . ■

Remark 4.24 :

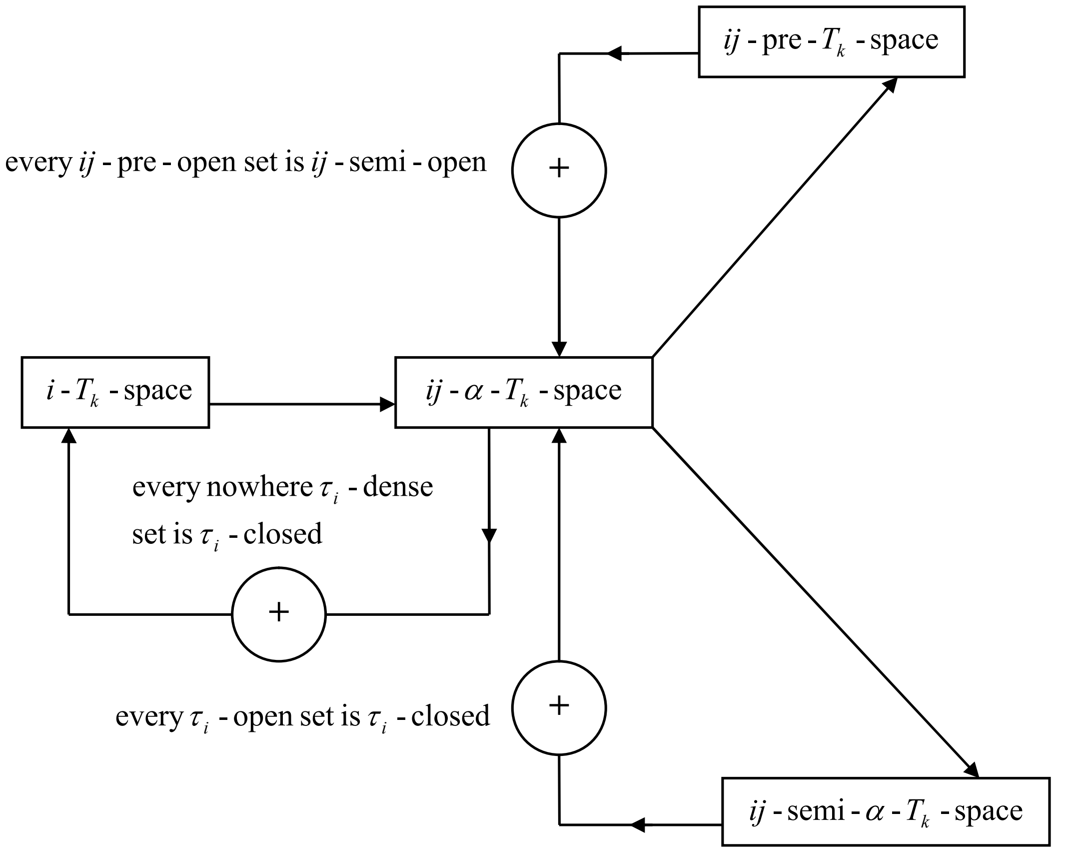
If every τ_i – open set is τ_i – closed and every nowhere τ_i – dense set is τ_i – closed in any bitopological space , then every ij - semi - α - T_k - space is i - T_k - space , where $k = 0,1,2,3,4$.

Proof :

Follows from remark (3.9) (ii) . ■

Remark 4.25 :

The following diagram shows the relations between i - T_k , ij - pre - T_k , ij - α - T_k and ij - semi - α - T_k spaces , where $k = 0,1,2,3,4$:



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