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# Semi - $\alpha$ - Connectedness in Bitopological Spaces

**By**

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## Abstract :

The objective of this paper is to study a special case of connectedness in bitopological spaces by considering $ij$-semi-$\alpha$-open sets and their relationships with $ij$-$\alpha$-connected space and $ij$-pre-connected space .

**Key words :** Bitopological space , $ij$ - semi - $\alpha$ - open set , $ij$ - semi- $\alpha$ - connected space .

## 1. Introduction :

The study of bitopological spaces was initiated by Kelly, J .C ., [5] . A triple $(X,\tau_1,\tau_2)$ is called bitopological space if $(X,\tau_1)$ and $(X,\tau_2)$ are two topological spaces . In 1997 , Kumar Sampath , S ., [6] introduced the concept of $ij$-$\alpha$-open sets in bitopological spaces . In 1981 , Bose , S. , [1] introduced the notion of $ij$-semi-open sets in bitopological spaces . In 1992 , Kar A ., [4] have introduced the notion of $ij$-pre-open sets in bitopological spaces . In 2012 , H . I . Al-Rubaye , Qaye , [2] introduced the notion of $ij$-semi-$\alpha$-open sets in bitopological spaces .

In this paper, we study especial case of connectedness in bitopological spaces by considering $ij$-semi-$\alpha$-open sets , we prove some results about them comparing with similar cases in topological spaces .

## 2. Preliminaries :

Throughout the paper , spaces always mean a bitopological spaces , the closure and the interior of any subset $A$ of $X$ with respect to $\tau_i$, will be denoted by $\tau_i - cl(A)$ , and $\tau_i - \text{int}(A)$ respectively, for $i = 1,2$ .

**Definition 2.1 :**

Let $(X,\tau_1,\tau_2)$ be a bitopological space , $A \subseteq X$ , $A$ is said to be :

(i) $ij$ - pre - open set [4] if $A \subseteq i-\text{int}(j-cl(A))$, where $i \neq j; i, j = 1,2$,

(ii) $ij$ - semi - open set [1] if $A \subseteq j-cl(i-\text{int}(A))$, where $i \neq j; i, j = 1,2$,

(iii) $ij$ - $\alpha$ - open set [6] if $A \subseteq i-\text{int}(j-cl(i-\text{int}(A)))$, where $i \neq j; i, j = 1,2$.

**Remark 2.2 :**

The family of $ij$ - pre - open ( resp. $ij$ - semi - open and $ij$ - $\alpha$ - open ) sets of $X$ is denoted by $ij$ - $PO(X)$ ( resp. $ij$ - $SO(X)$ and $ij$ - $\alpha O(X)$ ) , where $i \neq j; i, j = 1,2$.

**Example 2.3 :**

Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\}\}$ , and $\tau_2 = \{X,\phi,\{b\},\{c\},\{b,c\}\}$ .

$(X,\tau_1)$ and $(X,\tau_2)$ are two topological spaces , then $(X,\tau_1,\tau_2)$ is a bitopological space .

The family of all 12 - pre - open sets of $X$ is : 12 - $PO(X) = \{X,\phi,\{a\},\{b,c\}\}$ .

The family of all 12 - semi - open sets of $X$ is : 12 - $SO(X) = \{X,\phi,\{a\}\}$ .

The family of all 12 - $\alpha$ - open sets of $X$ is : 12 - $\alpha O(X) = \{X,\phi,\{a\}\}$ .

**Definition 2.4 :**

The complement of an $ij$ - pre - open ( resp. $ij$ - semi - open and $ij$ - $\alpha$ - open ) set is said to be $ij$ - pre - closed ( resp. $ij$ - semi - closed and $ij$ - $\alpha$ - closed ) set . The family of $ij$ - pre - closed ( resp. $ij$ - semi - closed and $ij$ - $\alpha$ - closed ) sets of $X$ is denoted by $ij$ - $PC(X)$ ( resp. $ij$ - $SC(X)$ and $ij$ - $\alpha C(X)$ ) , where $i \neq j; i, j = 1,2$.

**Remark 2.5 :**

It is clear by definition that in any bitopological space the following hold :

(i) every $\tau_i$ - open set is $ij$ - pre - open , $ij$ - semi - open , $ij$ - $\alpha$ - open set .

(ii) every $ij$ - $\alpha$ - open set is $ij$ - pre - open , $ij$ - semi - open set .

(iii) the concept of $ij$ - pre - open and $ij$ - semi - open sets are independent .

**Proposition 2.6 :**

A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is $ij$ - $\alpha$ - open set if and only if there exists an $\tau_i$ - open set $U$ , such that $U \subseteq A \subseteq i-\text{int}(j-cl(U))$ .

**Proof :**

This follows directly from the definition (2.1) (iii) . ■

**Proposition 2.7 : [7]**

A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is $ij$-semi-open set if and only if there exists an $\tau_i$ – open set $U$, such that $U \subseteq A \subseteq j-cl(U)$ .

**Proposition 2.8 : [3]**

A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is $ij$-pre-open set if and only if there exists an $\tau_i$ – open set $U$, such that $A \subseteq U \subseteq j-cl(A)$ .

**Theorem 2.9 :**

A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is an $ij$-$\alpha$-open set if and only if $A$ is $ij$-semi-open set and $ij$-pre-open set .

**Proof :**

Follows from definition (2.1) and remark (2.5) . ■

**Definition 2.10 : [4,6]**

Let $(X,\tau_1,\tau_2)$ be a bitopological space and $A \subseteq X$ ,the intersection of all $ij$-$\alpha$-closed (resp. $ij$-pre-closed ) sets containing $A$ is called $ij$-$\alpha$-closure (resp. $ij$-pre-closure ) of $A$ , and is denoted by $ij\text{-}\alpha\text{-}cl(A)$ (resp. $ij\text{-pre-}cl(A)$ ) ;

i.e $ij\text{-}\alpha\text{-}cl(A) = \bigcap\{B \subseteq X : B \text{ is } ij\text{-}\alpha\text{-closed set}, A \subseteq X\}$ and

$ij\text{-pre-}cl(A) = \bigcap\{B \subseteq X : B \text{ is } ij\text{-pre-closed set}, A \subseteq X\}$ .

**Definition 2.11 : [2]**

Let $(X,\tau_1,\tau_2)$ be a bitopological space , $A \subseteq X$ . Then $A$ is a said to be $ij$-semi-$\alpha$-open set if there exists an $ij$-$\alpha$-open set $U$ in $X$ , such that $U \subseteq A \subseteq j-cl(U)$. The family of all $ij$-semi-$\alpha$-open sets of $X$ is denoted by $ij\text{-}S_\alpha O(X)$, where $i \neq j; i, j = 1,2$ .

The following proposition will give an equivalent definition of $ij$-semi-$\alpha$-open sets .

**Proposition 2.12 : [2]**

Let $(X,\tau_1,\tau_2)$ be a bitopological space , $A \subseteq X$ . Then $A$ is an $ij$-semi-$\alpha$-open set if and only if $A \subseteq j-cl(i-\text{int}(j-cl(i-\text{int}(A))))$ .

**Remark 2.13 : [2]**

The intersection of any two $ij$-semi-$\alpha$-open sets is not necessary $ij$-semi-$\alpha$-open set as in the following example .

**Example 2.14 :**

Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$, and $\tau_2=\{X,\phi,\{a\}\}$ . The family of all 12-semi-$\alpha$-open sets of $X$ is : $12\text{-}S_{\alpha}O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$ .

Hence $\{a,c\}$ and $\{b,c\}$ are two 12-semi-$\alpha$-open sets, but $\{a,c\}\cap\{b,c\}=\{c\}$ is not 12-semi-$\alpha$-open set.

**Proposition 2.15 : [2]**

The union of any family of $ij$-semi-$\alpha$-open sets is $ij$-semi-$\alpha$-open set .

**Remark 2.16 : [2]**

(i) Every $\tau_i$ – open set is $ij$-semi-$\alpha$-open set ,but the converse need not be true .

(ii) If every $\tau_i$ – open set is $\tau_i$ – closed and every nowhere $\tau_i$ – dense set is $\tau_i$ – closed in any bitopological space , then every $ij$-semi-$\alpha$-open set is an $\tau_i$ – open set .

**Remark 2.17 : [2]**

(i) Every $ij$-$\alpha$-open set is $ij$-semi-$\alpha$-open set , but the converse is not true in general .

(ii) If every $\tau_i$ – open set is $\tau_i$ – closed set in any bitopological space , then every $ij$-semi-$\alpha$-open set is an $ij$-$\alpha$-open set .

**Remark 2.18 : [2]**

The concepts of $ij$-semi-$\alpha$-open and $ij$-pre-open sets are independent , as the following example .

**Example 2.19 :**

In example (2.3) , $\{b,c\}$ is a 12-pre-open set but not 12-semi-$\alpha$-open set.

**Remark 2.20 : [2]**

(i) It is clear that every $ij$-semi-open and $ij$-pre-open subsets of any bitopological space is $ij$-semi-$\alpha$-open set ( by theorem (2.9) and remark (2.17) (i) ) .

(ii) An $ij$-semi-$\alpha$-open set in any bitopological space $(X,\tau_1,\tau_2)$ is $ij$-pre-open set if every $\tau_i$-open subset of $X$ is $\tau_i$-closed set ( from remark (2.17) (ii) and remark (2.5) (iii) ) .

**Definition 2.21 : [2]**

The complement of $ij$-semi-$\alpha$-open set is called $ij$-semi-$\alpha$-closed set . Then family of all $ij$-semi-$\alpha$-closed sets of $X$ is denoted by $ij\text{-}S_\alpha C(X)$, where $i \neq j; i, j = 1,2$.

**Remark 2.22 : [2]**

The intersection of any family of $ij$-semi-$\alpha$-closed sets is $ij$-semi-$\alpha$-closed set .

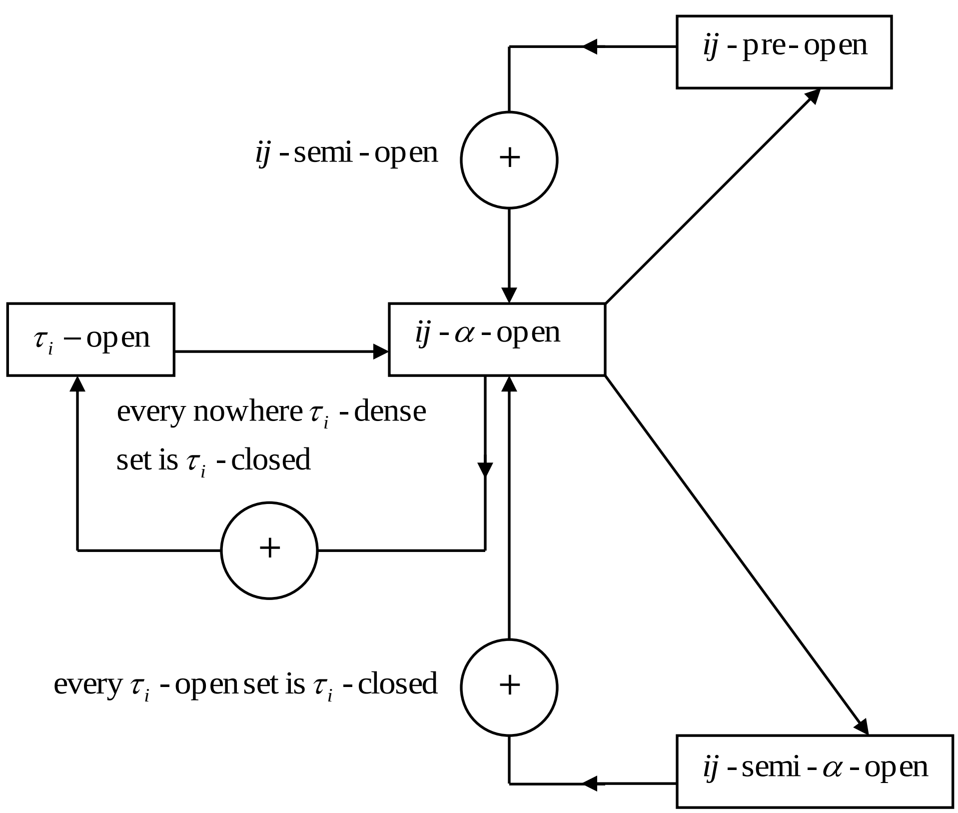
**Definition 3.17 : [2]**

Let $(X,\tau_1,\tau_2)$ be a bitopological space and $A \subseteq X$ ,the intersection of all $ij$-semi-$\alpha$-closed sets containing $A$ is called $ij$-semi-$\alpha$-closure of $A$ , and is denoted by $ij\text{-}S_\alpha\text{-}cl(A)$ ;

i.e $ij\text{-}S_\alpha\text{-}cl(A) = \bigcap\{B \subseteq X : B \text{ is } ij\text{-semi-}\alpha\text{-closed set}, A \subseteq X\}$.

**Remark 2.23 : [2]**

The following diagram shows the relations among the different types of weakly open sets that were studied in this section :



## 3 . *ij* - Semi - $\alpha$ - Connectedness in Bitopological Spaces :

In this section the notion of $ij$ - semi - $\alpha$ - connected space is introduced in bitopological spaces and their relationships with $ij$ - $\alpha$ - connected space and $ij$ - pre - connected space are studied .

**Definition 3.1 :**

Let $(X, \tau_1, \tau_2)$ be a bitopological space , two non-empty subsets $A$ and $B$ of $X$ are said to be $ij$ - semi - $\alpha$ - separated if $A \cap ij\text{-}S_\alpha\text{-}cl(B) = \phi$ and $ij\text{-}S_\alpha\text{-}cl(A) \cap B = \phi$ .

**Definition 3.2 :**

A bitopological space $(X, \tau_1, \tau_2)$ is called $ij$ - semi - $\alpha$ - connected if it is not the union of two non-empty $ij$ - semi - $\alpha$ - separated $ij$ - semi - $\alpha$ - open sets .

A subset $B \subseteq X$ is $ij$ - semi - $\alpha$ - connected if it is $ij$ - semi - $\alpha$ - connected as a subspace of $X$ .

An $ij$ - semi - $\alpha$ - disconnection of $X$ is a pair of complement , non-empty , $ij$ - semi - $\alpha$ - open $ij$ - semi - $\alpha$ - closed subsets .

**Remark 3.3 :**

The only $ij$ - semi - $\alpha$ - open $ij$ - semi - $\alpha$ - closed subsets in $ij$ - semi - $\alpha$ - connected space $X$ are $X$ and $\phi$ .

**Remark 3.4 :**

Every $ij$ - semi - $\alpha$ - connected space is $i$ - connected , but the converse is not true .

**Proof :**

Suppose that $X$ is not $i$ - connected , then $\exists$ two non-empty $A$ , $B$ are $i$ – open ∍ $A \cap B = \phi$ and $A \cup B = X$ . By remark ( 2.16 ) (i) , we have $A$ , $B$ are $ij$ - semi - $\alpha$ - open sets , $A \cup B = X$ and $A \cap B = \phi$ , hence $X$ is not $ij$ - semi - $\alpha$ - connected which is a contradiction . Thus , $X$ is $i$ - connected . ■

But the converse is not true as in the following example .

**Example 3.5 :**

Let $X=\{a,b,c\}$ , $\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$ , $\tau_2=\{X,\phi,\{a\},\{a,c\}\}$ .

The family of all 12 - semi - $\alpha$ - open sets of $X$ is : $12\text{-}S_\alpha O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$ .

Then $X$ is 1 - connected space , but $X$ is not 12 - semi - $\alpha$ - connected .

**Remark 3.6 :**

If every $\tau_i$ – open set is $\tau_i$ – closed and every nowhere $\tau_i$ – dense set is $\tau_i$ – closed in any bitopological space , then every $i$ - connected space is $ij$ - semi - $\alpha$ - connected .

**Proof :**

Follows from remark ( 2.16 ) (ii) . ■

**Definition 3.7 :**

A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $ij$ - semi - $\alpha$ - open if for each $\tau_i$ – open set $U$ of $X$ , $f(U)$ is $ij$ - semi - $\alpha$ - open in $Y$ .

**Definition 3.8 :**

A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $ij$ - semi - $\alpha$ - continuous if and only if the inverse image of each $i$ – open subset of $Y$ is $ij$ - semi - $\alpha$ - open subset of $X$ .

**Proposition 3.9 :**

Every $i$ – continuous function is $ij$ - semi - $\alpha$ - continuous .

**Proof :**

Follows from remark ( 2.16 ) (i) . ■

**Definition 3.10 :**

A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $ij$ - semi - $\alpha$ - irresolute if and only if the inverse image of each $ij$ - semi - $\alpha$ - open subset of $Y$ is $ij$ - semi - $\alpha$ - open subset of $X$ .

**Proposition 3.11 :**

Every $ij$ - semi - $\alpha$ - irresolute function is $ij$ - semi - $\alpha$ - continuous .

**Proof :**

Let $A$ be any $\sigma_i-$open set in $Y$. Then we have $A$ is an $ij$-semi-$\alpha$-open set in $Y$ [ from remark ( 2.16) (i) ] . Since $f$ is $ij$-semi-$\alpha$-irresolute function ,then $f^{-1}(A)$ is $ij$-semi-$\alpha$-open set in $X$ . Therefore $f$ is $ij$-semi-$\alpha$-continuous. ■

**Proposition 3.12 :**

Let $(X,\tau_1,\tau_2)$ and $(Y,\sigma_1,\sigma_2)$ be two bitopological spaces . If $X$ is $ij$-semi-$\alpha$-connected and $f$ is $ij$-semi-$\alpha$-continuous function from $(X,\tau_1,\tau_2)$ onto $(Y,\sigma_1,\sigma_2)$, then $Y$ is $i$-connected .

**Proof :**

Suppose that $A$ is an $i-$open $i-$closed subset of $Y$ , then $f^{-1}(A)$ is $ij$-semi-$\alpha$-open $ij$-semi-$\alpha$-closed in $X$ . Hence $f^{-1}(A)$ is $\phi$ or $X$ , but $X$ is $ij$-semi-$\alpha$-connected . So $A$ is $\phi$ or $Y$ . Hence $Y$ is $i$-connected . ■

**Proposition 3.13 :**

An $ij$-semi-$\alpha$-irresolute image of any $ij$-semi-$\alpha$-connected bitopological space is $ij$-semi-$\alpha$-connected .

**Proof :**

Follows directly from proposition ( 3.12 ) . ■

**Definition 3.14 :**

Let $(X,\tau_1,\tau_2)$ be a bitopological space , two non-empty subsets $A$ and $B$ of $X$ are said to be $ij$-$\alpha$-separated ( resp. $ij$-pre-separated) if $A\cap ij\text{-}\alpha\text{-}cl(B)=\phi$ ( resp. $A\cap ij\text{-pre-}cl(B)=\phi$ ) and $ij\text{-}\alpha\text{-}cl(A)\cap B=\phi$ (resp . $ij\text{-pre-}cl(A)\cap B=\phi$ ) .

**Definition 3.15 :**

A bitopological space $(X,\tau_1,\tau_2)$ is called $ij$-$\alpha$-connected ( resp. $ij$-pre-connected ) space if it is not the union of two non-empty $ij$-$\alpha$-separated ( resp. $ij$-pre-separated ) $ij$-$\alpha$-open ( resp. $ij$-pre-open ) sets .

**Proposition 3.16 :**

Every $ij$ - pre - connected space is $ij$ - $\alpha$ - connected .

**Proof :**

Follows from remark (2.5) (ii) . ■

**Proposition 3.17 :**

If every $ij$ - pre - open set in a bitopological space $(X,\tau_1,\tau_2)$ is $ij$ - semi - open set , then $X$ is $ij$ - pre - connected space , whenever it is an $ij$ - $\alpha$ - connected space .

**Proof :**

Follows from theorem (2.9) . ■

**Proposition 3.18 :**

Every $ij$ - pre - connected space is $i$ - connected .

**Proof :**

Follows from remark (2.5) (i) . ■

**Remark 3.19 :**

Every $ij$ - semi - $\alpha$ - connected space is $ij$ - $\alpha$ - connected .

**Proof :**

Follows from remark (2.17) (i) . ■

**Proposition 3.20 :**

In a bitopological space $(X,\tau_1,\tau_2)$ , if every $\tau_i$ – open subset of $X$ is $\tau_i$ – closed set , then $X$ is $ij$ - semi - $\alpha$ - connected space , whenever it is an $ij$ - $\alpha$ - connected space .

**Proof :**

Follows from remark (2.17) (ii) . ■

**Remark 3.21 :**

The concepts of $ij$ - pre - connected space and $ij$ - semi - $\alpha$ - connected space are independent .

**Proposition 3.22 :**

If every $\tau_i$ – open set in a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_i$ – closed, then $X$ is $ij$-semi-$\alpha$-connected, whenever it is $ij$-pre-connected.

**Proof :**

Follows from propositions (3.16) and (3.20). ■

**Proposition 3.23 :**

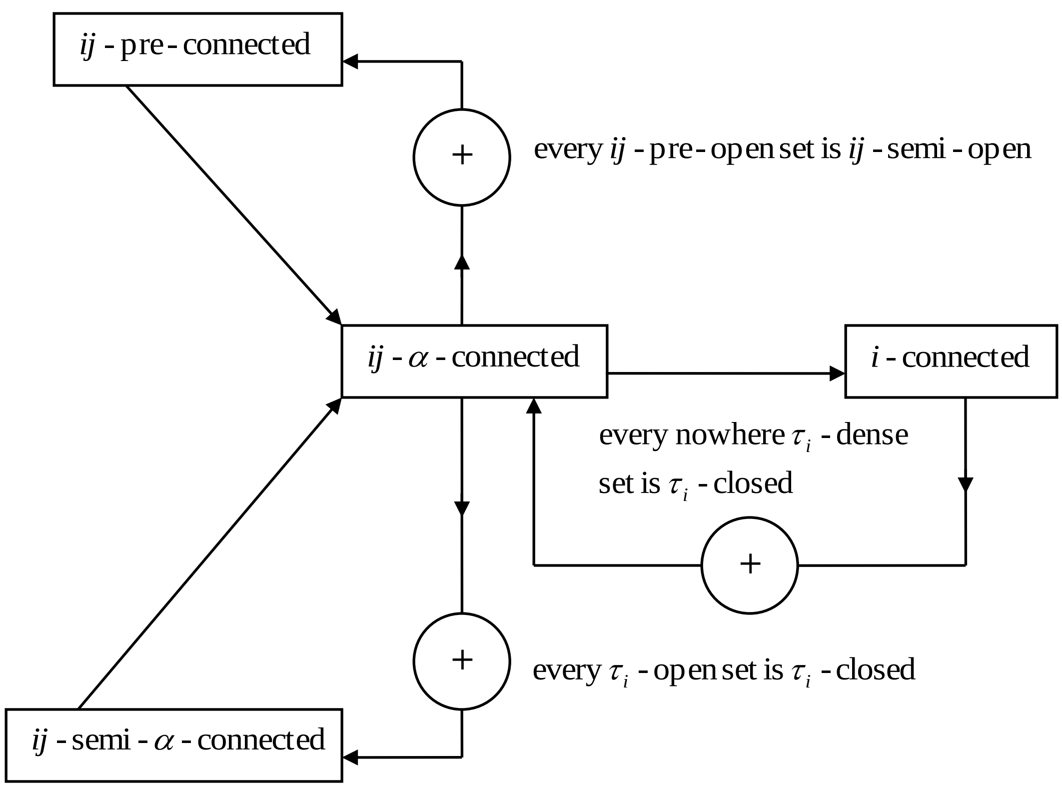
If every $ij$-pre-open set in a bitopological space $(X, \tau_1, \tau_2)$ is $ij$-semi-open set, then $X$ is $ij$-pre-connected space, whenever it is an $ij$-semi-$\alpha$-connected space.

**Proof :**

Follows from remark (3.19) and proposition (3.20). ■

**Remark 3.24 :**

The following diagram shows the relations among the different types of connectedness :



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