Semi - α - Compactness in Bitopological Spaces

By

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Abstract :

In this paper, we study especial case of compactness in bitopological spaces by considering ij - semi -α - open sets , we prove some results about them comparing with similar cases in topological spaces .

Key words : Bitopological space , ij - semi - α - open set, ij - semi - α - compact space .

الخلاصة :

في هذا البحث تم دراسة حالة خاصة من التراص في الفضاءات ثنائية التبولوجي من خلال الاعتماد على مجموعات ij - شبه ألفا المفتوحة ، أثبتنا بعض النتائج عنها مقارنة مع حالات مماثلة في الفضاءات التبولوجية .

1. Introduction :

The study of bitopological spaces was initiated by [Kelly, J .C ., 1963] . A triple (X,τ_1,τ_2) is called bitopological space if (X,τ_1) and (X,τ_2) are two topological spaces . In 1990 ,Jelic , M ., introduced the concept of ij - α - open sets in bitopological spaces . In 1981 , Bose ,S. ,introduced the notion of ij - semi - open sets in bitopological spaces . In 1992 , Kar A . , have introduced the notion of ij - pre - open sets in bitopological spaces . In 2012 , H . I . Al-Rubaye , Qaye , introduced the notion of ij - semi - α - open sets in bitopological spaces .

In this paper , we study especial cases of compactness in bitopological spaces by considering ij - semi - α - open sets and their relationships with ij - α - compact space and ij - pre - compact space are studied .

2. Preliminaries :

Throughout the paper , spaces always mean a bitopological spaces , the closure and the interior of any subset A of X with respect to τ_i, will be denoted by $\tau_i - cl(A)$, and $\tau_i - \text{int}(A)$ respectively, for $i = 1,2$.

Definition 2.1 :

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$, A is said to be :

(i) ij - pre - open set [Kar A . , 1992] if $A \subseteq i-\text{int}(j-cl(A))$, where $i \neq j; i, j = 1,2$,

(ii) ij - semi - open set [Bose , S . , 1981] if $A \subseteq j-cl(i-\text{int}(A))$, where $i \neq j; i, j = 1,2$,

(iii) ij - α - open set [Kumar Sampath , S. , 1997] if $A \subseteq i-\text{int}(j-cl(i-\text{int}(A)))$, where $i \neq j; i, j = 1,2$.

Remark 2.2 :

The family of ij - pre - open (resp. ij - semi - open and ij - α - open) sets of X is denoted by ij - $PO(X)$ (resp. ij - $SO(X)$ and ij - $\alpha O(X)$) , where $i \neq j; i, j = 1,2$.

Example 2.3 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\}\}$, and $\tau_2 = \{X,\phi,\{b\},\{c\},\{b,c\}\}$.
(X,τ_1) and (X,τ_2) are two topological spaces , then (X,τ_1,τ_2) is a bitopological space .
The family of all 12 - pre - open sets of X is : 12 - $PO(X) = \{X,\phi,\{a\},\{b,c\}\}$.
The family of all 12 - semi - open sets of X is : 12 - $SO(X) = \{X,\phi,\{a\}\}$.
The family of all 12 - α - open sets of X is : 12 - $\alpha O(X) = \{X,\phi,\{a\}\}$.

Definition 2.4 :

The complement of an ij - pre - open (resp. ij - semi - open and ij - α - open) set is said to be ij - pre - closed (resp. ij - semi - closed and ij - α - closed) set . The family of ij - pre - closed (resp. ij - semi - closed and ij - α - closed) sets of X is denoted by ij - $PC(X)$ (resp. ij - $SC(X)$ and ij - $\alpha C(X)$) , where $i \neq j; i, j = 1,2$.

Remark 2.5 :

It is clear by definition that in any bitopological space the following hold :

(i) every τ_i – open set is ij - pre - open , ij - semi - open , ij - α - open set .

(ii) every ij - α - open set is ij - pre - open , ij - semi - open set .

(iii) the concept of ij - pre - open and ij - semi - open sets are independent .

Proposition 2.6 :

A subset A of a bitopological space (X,τ_1,τ_2) is ij - α - open set if and only if there exists an τ_i – open set U, such that $U \subseteq A \subseteq i-\text{int}(j-cl(U))$.

Proof :

This follows directly from the definition (2.1) (iii) . ■

Proposition 2.7 : [Maheshwari , S . N . and Prasad , R . ,1977]

A subset A of a bitopological space (X,τ_1,τ_2) is ij - semi - open set if and only if there exists an τ_i – open set U, such that $U \subseteq A \subseteq j-cl(U)$.

Proposition 2.8 : [Jelic , M . , 1990]

A subset A of a bitopological space (X,τ_1,τ_2) is ij - pre - open set if and only if there exists an τ_i – open set U, such that $A \subseteq U \subseteq j-cl(A)$.

Theorem 2.9 :

A subset A of a bitopological space (X,τ_1,τ_2) is an ij - α - open set if and only if A is ij - semi - open set and ij - pre - open set .

Proof :

Follows from definition (2.1) and remark (2.5) . ■

Definition 2.10 : [H . I. Al-Rubaye , Qaye , 2012]

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$. Then A is a said to be ij - semi - α - open set if there exists an ij - α - open set U in X , such that $U \subseteq A \subseteq j-cl(U)$. The family of all ij - semi - α - open sets of X is denoted by ij-$S_\alpha O(X)$, where $i \neq j; i, j = 1,2$.

Example 2.11 :

Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{a,b\}\}$, and $\tau_2=\{X,\phi,\{b\},\{c\},\{b,c\}\}$. (X,τ_1) and (X,τ_2) are two topological spaces , then (X,τ_1,τ_2) is a bitopological space . The family of all 12 - semi - α - open sets of X is : 12 - $S_\alpha O(X)=\{X,\phi,\{a\},\{a,b\}\}$.

The family of all 21 - semi - α - open sets of X is : 21 - $S_\alpha O(X) = \{X, \phi, \{b\}, \{c\}, \{b,c\}\}$.

The following proposition will give an equivalent definition of ij - semi - α - open sets .

Proposition 2.12 : [H . I. Al-Rubaye , Qaye , 2012]

Let (X, τ_1, τ_2) be a bitopological space , $A \subseteq X$. Then A is an ij - semi - α - open set if and only if $A \subseteq j - cl(i - \text{int}(j - cl(i - \text{int}(A))))$.

Remark 2.13 : [H . I. Al-Rubaye , Qaye , 2012]

The intersection of any two ij - semi - α - open sets is not necessary ij - semi - α - open set as in the following example .

Example 2.14 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$, and $\tau_2 = \{X, \phi, \{a\}\}$. The family of all 12 - semi - α - open sets of X is : 12 - $S_\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.

Hence $\{a,c\}$ and $\{b,c\}$ are two 12 - semi - α - open sets , but $\{a,c\} \cap \{b,c\} = \{c\}$ is not 12 - semi - α - open set .

Proposition 2.15 : [H . I. Al-Rubaye , Qaye , 2012]

The union of any family of ij - semi - α - open sets is ij - semi - α - open set .

Remark 2.16 : [H . I. Al-Rubaye , Qaye , 2012]

(i) Every $\tau_i -$ open set is ij - semi - α - open set ,but the converse need not be true .

(ii) If every $\tau_i -$ open set is $\tau_i -$ closed and every nowhere $\tau_i -$ dense set is $\tau_i -$ closed in any bitopological space , then every ij - semi - α - open set is an $\tau_i -$ open set .

Remark 2.17 : [H . I. Al-Rubaye , Qaye , 2012]

(i) Every ij - α - open set is ij - semi - α - open set , but the converse is not true in general .

(ii) If every τ_i – open set is τ_i – closed set in any bitopological space , then every ij - semi - α - open set is an ij - α - open set .

Remark 2.18 : [H . I. Al-Rubaye , Qaye , 2012]

The concepts of ij - semi - α - open and ij - pre - open sets are independent , as the following example .

Example 2.19 :

In example (2.3) , $\{b,c\}$ is a 12 - pre - open set but not 12 - semi - α - open set .

Remark 2.20 : [H . I. Al-Rubaye , Qaye , 2012]

(i) It is clear that every ij - semi - open and ij - pre - open subsets of any bitopological space is ij - semi - α - open set (by theorem (2.9) and remark (2.17) (i)) .

(ii) An ij - semi - α - open set in any bitopological space (X,τ_1,τ_2) is ij - pre - open set if every τ_i – open subset of X is τ_i – closed set (from remark (2.17)(ii) and remark (2.5)(iii)) .

Definition 2.21 : [H . I. Al-Rubaye , Qaye , 2012]

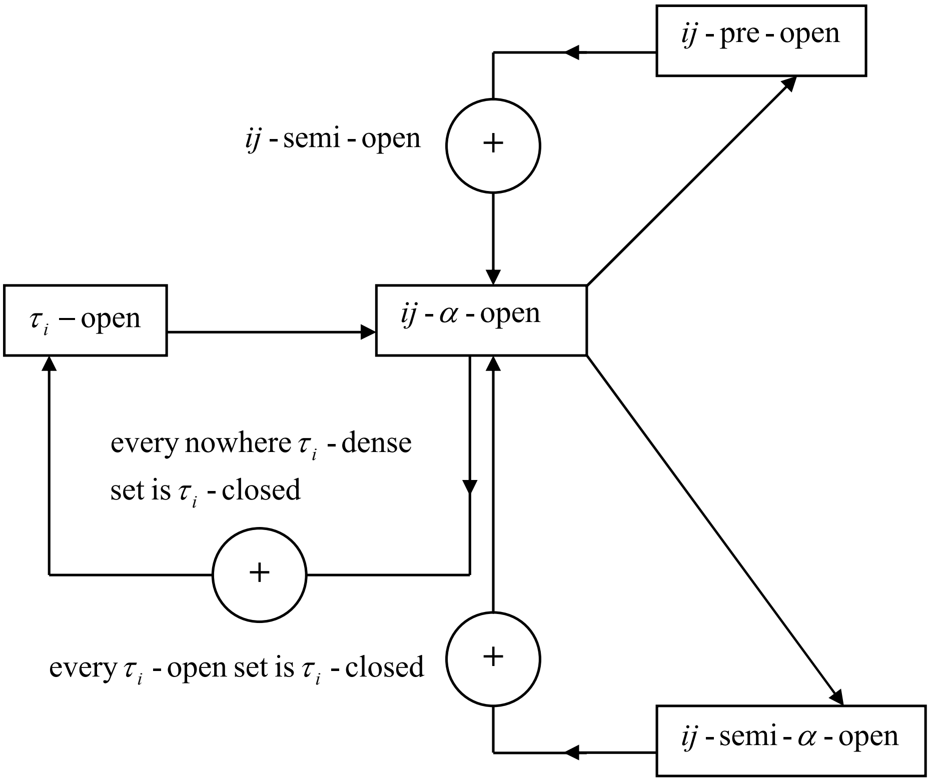
The complement of ij - semi - α - open set is called ij - semi - α - closed set. Then family of all ij - semi - α - closed sets of X is denoted by ij - $S_\alpha C(X)$, where $i \neq j; i, j = 1,2$.

Remark 2.22 : [H . I. Al-Rubaye , Qaye , 2012]

The intersection of any family of ij - semi - α - closed sets is ij - semi - α - closed set .

Remark 2.23 : [H . I. Al-Rubaye , Qaye , 2012]

The following diagram shows the relations among the different types of weakly open sets that were studied in this section :



3 . *ij* - Semi - α - Compactness in Bitopological Spaces :

In this section the notion of ij - semi - α - compact space is introduced in bitopological spaces and their relationships with ij - α - compact space and ij - pre - compact space are studied .

Definition 3.1 :

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$, a family W of subsets of X is said to be ij - semi - α - open cover of A if and only if W covers A and W is a subfamily of $ij\text{-}S_\alpha O(X)$.

Definition 3.2 :

A bitopological space (X,τ_1,τ_2) is said to be ij - semi - α - compact if and only if every ij - semi - α - open cover of X has a finite subcover .

Remark 3.3 :

Every ij - semi - α - compact space is i - compact .

Proof :

Follows from remark (2.16) (i) . ■

Remark 3.4 :

If every τ_i – open set is τ_i – closed and every nowhere τ_i – dense set is τ_i – closed in any bitopological space , then every i - compact space is ij - semi - α - compact .

Proof :

Follows from remark (2.16) (ii) . ■

Proposition 3.5 :

Every ij - semi - α - closed subset of ij - semi - α - compact space is ij - semi - α - compact .

Proof :

Let (X,τ_1,τ_2) be ij - semi - α - compact space , and let A be ij - semi - α - closed subset of X . Let $W = \{U_\lambda : \lambda \in \Lambda\}$ be ij - semi - α - open cover of A .Since A is ij - semi - α - closed subset of X, then $X - A$ is ij - semi - α - open subset of X . Therefore , the family $\{U_\lambda : \lambda \in \Lambda\} \bigcup \{X - A\}$ is ij - semi - α - open cover of X , and since (X,τ_1,τ_2) is ij - semi - α - compact , it has a finite subcover . As $A \subseteq X$ and $X - A$ covers no part of A , a finite number of members of W , say $U_{\lambda 1}, U_{\lambda 2}, \ldots\ldots, U_{\lambda n}$, have the property that $A \subseteq \bigcup_{i=1}^{n} U_{\lambda i}$. Hence A is ij - semi - α - compact space . ■

Proposition 3.6 :

If A and B are two ij - semi - α - compact subsets of a bitopological space (X, τ_1, τ_2) , then $A \bigcup B$ is ij - semi - α - compact .

Proof :

Follows from proposition (2.15) . ■

Definition 3.7 :

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function , then f is called :

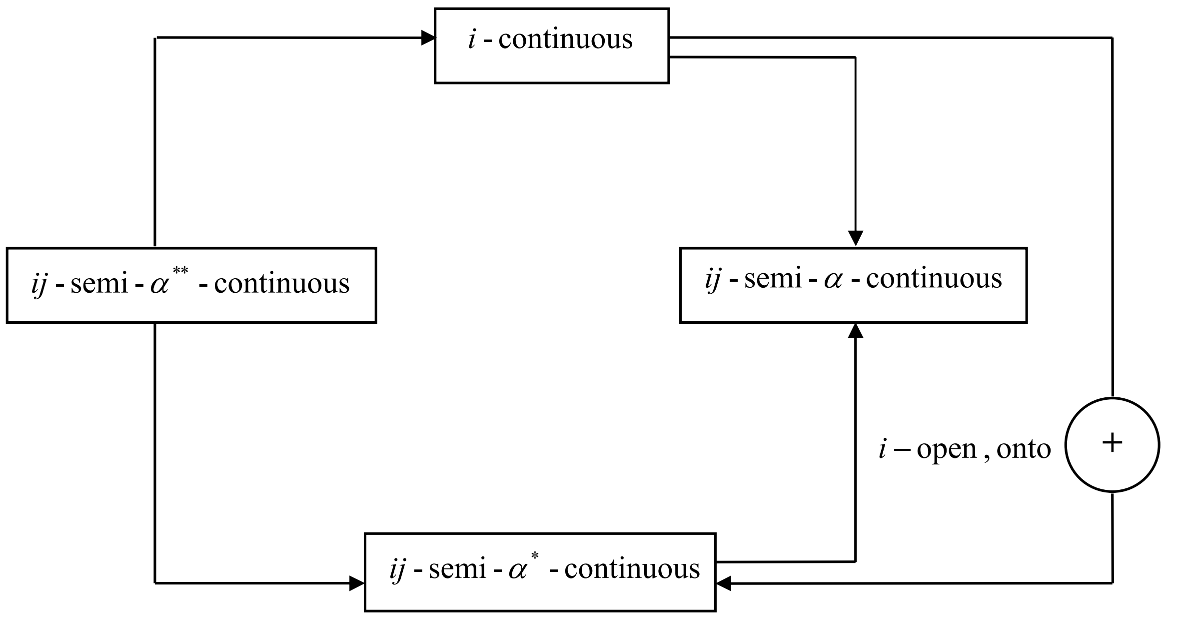
(i) ij - semi - α - continuous if and only if the inverse image of each $i-$open subset of Y is ij - semi - α - open subset of X .

(ii) ij - semi - α^* - continuous if and only if the inverse image of each ij - semi - α - open subset of Y is ij - semi - α - open subset of X .

(iii) ij - semi - α^{**} - continuous if and only if the inverse image of each ij - semi - α - open subset of Y is an $i-$open subset of X .

Remark 3.8 :

The following diagram explain the relationships among the different types of weakly continuous function :



Theorem 3.9 :

The ij - semi - α - continuous image of ij - semi - α - compact space is i - compact .

Proof :

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an ij - semi - α - continuous, onto function and let (X,τ_1,τ_2) be ij - semi - α - compact space . To prove that Y is i - compact .

Let $\{U_\lambda:\lambda\in\Lambda\}$ be an i – open cover of Y , implies $\{f^{-1}(U_\lambda):\lambda\in\Lambda\}$ is ij - semi - α - open cover of X , which is ij - semi - α - compact space . So , there exist $\lambda_1,\lambda_2,........,\lambda_n\in\Lambda$, such that $\{f^{-1}(U_{\lambda i}):i=1,2,.....,n\}$ is a finite subcover of X . Implies $\{U_{\lambda i}:i=1,2,.....,n\}$ is a finite subcover of Y . So Y is i - compact . ■

Corollary 3.10 :

The i - continuous image of ij - semi - α - compact space is i - compact .

Proof :

Since every i - continuous function is ij - semi - α - continuous, theorem (3.9) is applicable . ■

Theorem 3.11 :

The ij - semi - α^* - continuous image of ij - semi - α - compact space is ij - semi - α - compact .

Proof :

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an ij - semi - α^* - continuous , onto function and let (X,τ_1,τ_2) be ij - semi - α - compact space . To prove that (Y,σ_1,σ_2) is ij - semi - α - compact . Let $\{U_\lambda:\lambda\in\Lambda\}$ be ij - semi - α - open cover of Y , then $\{f^{-1}(U_\lambda):\lambda\in\Lambda\}$ is ij - semi - α - open cover of X , and since (X,τ_1,τ_2) is ij - semi - α - compact space , then there exist an indices $\lambda_1,\lambda_2,........,\lambda_n\in\Lambda$, such that $\{f^{-1}(U_{\lambda i}):i=1,2,.....,n\}$ is a finite subcover of X . Since f is onto , $\{U_{\lambda i}:i=1,2,.....,n\}$ is a finite subcover of Y . So Y is ij - semi - α - compact . ■

Corollary 3.12 :

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an $i-$open , i - continuous function of X onto Y . If (X,τ_1,τ_2) is ij - semi - α - compact , then (Y,σ_1,σ_2) is ij - semi - α - compact .

Proof :

Since every $i-$open , onto , i - continuous function is ij - semi - α^* - continuous . Theorem (3.11) is applicable . ■

Corollary 3.13 :

Let $\{X_\lambda:\lambda\in\Lambda\}$ be any family of bitopological spaces . If the product space $\prod_{\lambda\in\Lambda}X_\lambda$ is ij - semi - α - compact , then X_λ is ij - semi - α - compact , for each $\lambda\in\Lambda$.

Proof :

Since each projection $\rho_\lambda:\prod_{\lambda\in\Lambda}X_\lambda\to X_\lambda$ is i - continuous , $i-$open function of $\prod_{\lambda\in\Lambda}X_\lambda$ onto X_λ . It follows , by corollary (3.12) , that each X_λ is ij - semi - α - compact . ■

Corollary 3.14 :

The ij - semi - α^* - continuous image of ij - semi - α - compact space is i - compact .

Proof :

Follows from theorem (3.11) and remark (3.3) . ■

Theorem 3.15 :

The ij - semi - α^{**} - continuous image of i - compact space is ij - semi - α - compact .

Proof :

Let (X,τ_1,τ_2) be an i - compact space , and $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an ij - semi - α^{**} - continuous , onto function . To prove Y is ij - semi - α - compact space .

Let $\{U_\lambda:\lambda\in\Lambda\}$ be ij - semi - α - open cover of Y , then $\{f^{-1}(U_\lambda):\lambda\in\Lambda\}$ is an $i-$open cover of X , and so can be reduced to a finite subcover $f^{-1}(U_{\lambda 1}),f^{-1}(U_{\lambda 2}),\ldots\ldots,f^{-1}(U_{\lambda n})$; it is evident that $U_{\lambda 1},U_{\lambda 2},\ldots\ldots,U_{\lambda n}$ is a finite subcover of Y . So Y is ij - semi - α - compact . ■

Corollary 3.16 :

The ij - semi - α^{**} - continuous image of i - compact space is i - compact .

Proof :

Follows from theorem (3.15) and remark (3.3) . ■

Definition 3.17 :

Let (X,τ_1,τ_2) be a bitopological space , $A\subseteq X$, a family W of subsets of X is said to be ij - α - open (resp. ij - pre - open) cover of A if and only if W covers A and W is a subfamily of ij - $\alpha O(X)$ (resp. ij - $PO(X)$) .

Definition 3.18 :

A bitopological space (X,τ_1,τ_2) is said to be ij - α - compact (resp. ij - pre - compact) space if and only if every ij - α - open (resp. ij - pre- open) cover of X has a finite subcover .

Proposition 3.19 :

Every ij - pre - compact space is ij - α - compact .

Proof :

Follows from remark (2.5) (ii) . ■

Proposition 3.20 :

If every ij - pre - open set in a bitopological space (X, τ_1, τ_2) is ij - semi - open set , then X is ij - pre - compact space , whenever it is an ij - α - compact space .

Proof :

Follows from theorem (2.9) . ■

Proposition 3.21 :

Every ij - pre - compact space is i - compact .

Proof :

Follows from remark (2.5) (i) . ■

Remark 3.22 :

Every ij - semi - α - compact space is ij - α - compact .

Proof :

Follows from remark (2.17) (i) . ■

Proposition 3.23 :

In a bitopological space (X, τ_1, τ_2) , if every τ_i – open subset of X is τ_i – closed set , then X is ij - semi - α - compact space , whenever it is an ij - α - compact space .

Proof :

Follows from remark (2.17) (ii) . ■

Remark 3.24 :

The concepts of ij - pre - compact space and ij - semi - α - compact space are independent .

Proposition 3.25 :

If every τ_i – open set in a bitopological space (X,τ_1,τ_2) is τ_i – closed, then X is ij - semi - α - compact , whenever it is ij - pre - compact .

Proof :

Follows from propositions (3.19) and (3.23) . ■

Proposition 3.26 :

If every ij - pre - open set in a bitopological space (X,τ_1,τ_2) is ij - semi - open set , then X is ij - pre - compact space , whenever it is an ij - semi - α - compact space .

Proof :

Follows from remark (3.22) and proposition (3.23) . ■

Corollary 3.27 :

The ij - semi - α^{**} - continuous image of i - compact space is ij - α - compact .

Proof :

Follows from theorem (3.15) and remark (3.22) . ■

Corollary 3.28 :

The ij - semi - α^{**} - continuous image of ij - α - compact space is ij - semi - α - compact (ij - α - compact , i - compact , respectively) .

Proof :

Clear . ■

Corollary 3.29 :

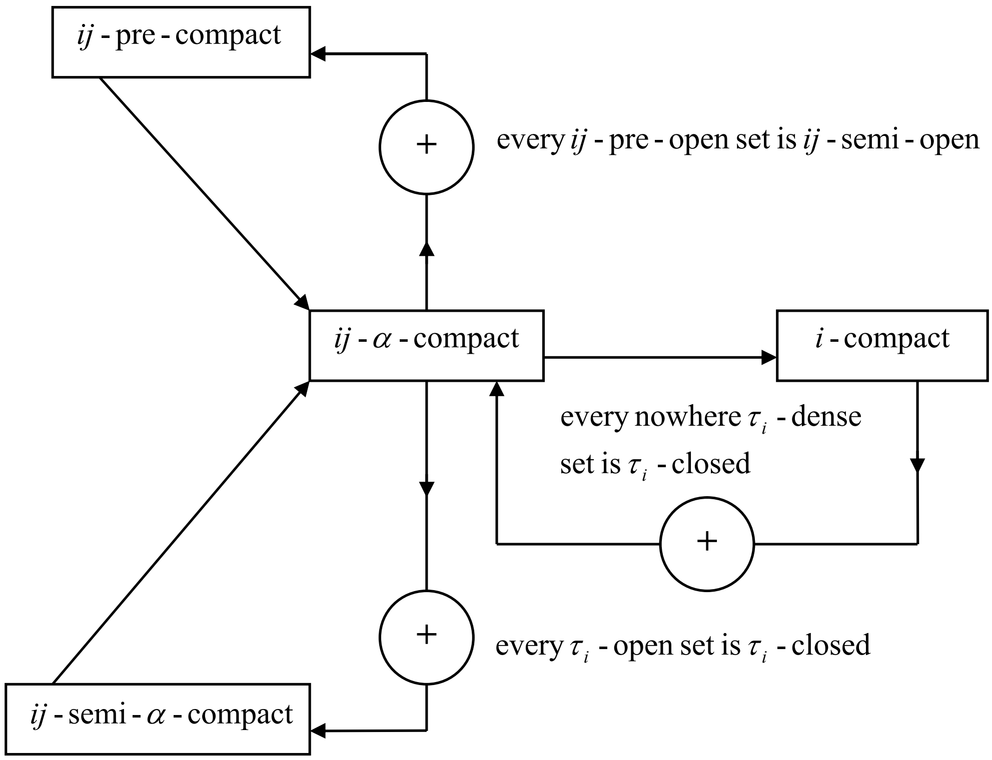
The ij - semi - α^{**} - continuous image of ij - semi - α - compact space is ij - semi - α - compact (ij - α - compact , i - compact , respectively) .

Proof :

Clear . ■

Remark 3.30 :

The following diagram shows the relations among the different types of compactness :



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