Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 20, 987 - 993

Some Applications of Fractional Calculus of a Class of *k* – Uniformly Convex Functions and Related Class of *k* – Starlike Functions Defined by Integral Operator I

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Abstract. In our paper, we study a class of univalent functions with negative coefficients defined by integral operator in the unit disk U by applications fractional calculus . We obtain some results for this class .

Mathematics Subject Classification: Primary 30C45

Keywords: Univalent Function , Fractional Calculus , Integral Operator , Integral representation

1. Introduction

Let R be the class of the functions defined by the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \quad (a_n \geq 0, n \in IN) \tag{1}$$

which are analytic and univalent in the unit disk $U = \{z \in C : |z| < 1\}$.
A function $f \in R$ is said to in the class $k - ST(\gamma)$, the class of $k-$starlike functions of order γ , $(0 \leq \gamma < 1)$ if f satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)}-1\right|+\gamma \quad , \quad k\ge 0 \ . \tag{2}$$

Replacing f in (2) by $zf'(z)$ we get the condition

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|+\gamma \quad , \quad k\ge 0 \ . \tag{3}$$

Required for the function f to be in the class $k-UCV(\gamma)$ of $k-$uniformly convex functions of order γ .

Uniformly starlike and convex functions were first introduced by Goodman [3] and then studied by various authors like Rania and Bapna [5],Srivastava [7], Khairnar and Mena More [4] , Atshan and Buti [1].

Lemma 1 : [2]
The function $f(z)$ defined by (1) is inn the class $k-UCV(\lambda,\gamma,\beta,\delta)$ if and only if

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n \le 1-\gamma \tag{4}$$

where $0\le\gamma<1, k\ge 0, 0\le\lambda\le 1, \beta\ge 0$ and $\delta>-1$.

Lemma 2 : [2]
If $f\in k-UCV(\lambda,\gamma,\beta,\delta)$, then

$$a_n \le \frac{(1-\gamma)\Gamma(\beta+\delta+n)\Gamma(\delta+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(\beta+\delta+1)\Gamma(\delta+n)} \tag{5}$$

where $0\le\gamma<1, k\ge 0, 0\le\lambda\le 1, \beta\ge 0$ and $\delta>-1$.

Lemma 3 :
Let $f\in R$ be of the form (1) . If for some $k\ge 0$, the following inequality

$$\sum_{n=2}^{\infty} n\left[(k+1)(n-1)+(1-\gamma)\right]a_n \le 1-\gamma \ . \tag{6}$$

Holds true , then $f\in k-UCV(\gamma)$.

Proof :
It is suffices to show that

$$k\left|\frac{zf''(z)}{f'(z)}\right| - \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \le 1-\gamma$$. Notice that

$$k\left|\frac{zf''(z)}{f'(z)}\right| - \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \le (k+1)\left|\frac{zf''(z)}{f'(z)}\right|$$

$$\leq (k+1)\frac{\sum_{n=2}^{\infty} n(n-1)a_n}{1-\sum_{n=2}^{\infty} na_n}$$

The last inequality above is bounded above by $(1-\gamma)$, if

$\sum_{n=2}^{\infty} n[(k+1)(n-1)+(1-\gamma)]a_n \leq 1-\gamma$. This complete the proof .

Lemma 4 :
Let $f \in R$ be of the form (1) . If for some $k \geq 0$, the following inequality

$$\sum_{n=2}^{\infty} [(n-1)(k+1)+1]a_n \leq 1-\gamma \ . \qquad (7)$$

Holds true , then $f \in k-ST(\gamma)$.

Proof :
It is suffices to show that

$$k\left|\frac{zf'(z)}{f(z)}-1\right| - \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-1\right\} \leq 1-\gamma$$

Notice that

$$k\left|\frac{zf'(z)}{f(z)}-1\right| - \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-1\right\} \leq (k+1)\left|\frac{zf'(z)}{f(z)}-1\right|$$

$$\leq (k+1)\frac{\sum_{n=2}^{\infty} (n-1)a_n}{1-\sum_{n=2}^{\infty} a_n}$$

The last inequality above is bounded above by $(1-\gamma)$, if

$$\sum_{n=2}^{\infty} [(n-1)(k+1)+1]a_n \leq 1-\gamma \ .$$

This complete the proof .

3. Application of the fractional calculus :

Various operators of fractional calculus (that is, fractional derivative and fractional integral) have been rather extensively studied by many researcher (c.f. [8],[9],[10]). However , we try to restrict ourselves to the following definitions given by Owa[6] for convenience.

Definition 1:(Fractional integral operator). The fractional integral of order α is defined , for a function $f(z)$, by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt \qquad (\alpha > 0), \tag{8}$$

where $f(z)$ is an analytic function in a simply – connected region of the z – plane containing the origin, and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real , when $(z-t) > 0$.

Definition 2: (Fractional derivative operator). The fractional derivative of order α is defined , for a function $f(z)$ by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt \qquad (0 \le \alpha), \tag{9}$$

where $f(z)$ is constrained , and the multiplicity of $(z-t)^{-\alpha}$ is removed, as in Definition 2.

Definition 3: [Under the condition of Definition 3 the fractional derivative of order $k+\alpha$ $(k = 0,1,2,...)$ is defined by

$$D_z^{k+\alpha} f(z) = \frac{d^k}{dz^k} D_z^{\alpha} f(z), \qquad (0 \le \alpha < 1). \tag{10}$$

From definition 2 and 3 by applying a simple calculation, we get

$$G(z) = \Gamma(2+\alpha) z^{-\alpha} D_z^{-\alpha} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)} a_n z^n, \tag{11}$$

$$F(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(2-\alpha)\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n z^n. \tag{12}$$

In the next theorems , we show that the function $G(z)$ defined by (11) be in the classes $k-UCV(\gamma)$ and $k-ST(\gamma)$ respectively .

Theorem 1: If $f \in k-UCV(\lambda,\gamma,\beta,\delta)$ and for some $k \ge 0$ the following inequalities

$$\sum_{n=2}^{\infty} \frac{n(n-1)\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)[n(1+k)-(k+\gamma)]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \le \frac{\Gamma(\beta+\delta+1)(2-\gamma)}{(1-\gamma)(k+1)\Gamma(\delta+1)\Gamma(2+\alpha)} \tag{13}$$

and $$\sum_{n=2}^{\infty} \frac{n\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)[n(1+k)-(k+\gamma)]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \le \frac{\Gamma(\beta+\delta+1)}{(1-\gamma)^2\Gamma(\delta+1)\Gamma(2+\alpha)} \tag{14}$$

are hold true, then $G(z) \in k-UCV(\gamma)$.

Proof : From (11) and Lemma 2 , we get

$$\sum_{n=2}^{\infty} n\left[(k+1)(n-1)\right]\frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n$$

$$\le \frac{(1-\gamma)(k+1)\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)}\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \quad (15)$$

and

$$\sum_{n=2}^{\infty} n(1-\gamma)\frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n$$

$$\le \frac{(1-\gamma)^2\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)}\sum_{n=2}^{\infty}\frac{n\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} . \quad (16)$$

Hence , from (15) and (16) , we get

$$\sum_{n=2}^{\infty} n\left[(k+1)(n-1)+(1-\gamma)\right]\frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n$$

$$\le \frac{(1-\gamma)(k+1)\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)}\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)}$$

$$+\frac{(1-\gamma)^2\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)}\sum_{n=2}^{\infty}\frac{n\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \quad (17)$$

Finally , if we make use of the hypotheses (13) and (14) in (17) , we get

$$\sum_{n=2}^{\infty} n\left[(k+1)(n-1)+(1-\gamma)\right]\frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n \le 1-\gamma \ .$$

So by using (11) and Lemma 3 , we get $f \in k-UCV(\lambda,\gamma,\beta,\delta)$.
This complete the proof .

Theorem 2: If $f \in k-ST(\lambda,\gamma,\beta,\delta)$ and for some $k \ge 0$ the following inequalities

$$\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \le \frac{\Gamma(\beta+\delta+1)(2-\gamma)}{(1-\gamma)(k+1)\Gamma(\delta+1)\Gamma(2+\alpha)} \quad (18)$$

and

$$\sum_{n=2}^{\infty}\frac{\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \le \frac{\Gamma(\beta+\delta+1)}{(1-\gamma)\Gamma(\delta+1)\Gamma(2+\alpha)} \quad (19)$$

are hold true, then $G(z) \in k-ST(\gamma)$.

Proof : From (11) and Lemma 2 , we get

$$\sum_{n=2}^{\infty} \left[(k+1)(n-1)\right]\frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n$$

$$\le \frac{(1-\gamma)(k+1)\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)}\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \quad (20)$$

and

$$\sum_{n=2}^{\infty} \frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)} a_n$$

$$\leq \frac{(1-\gamma)\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \qquad (21)$$

Hence , from (20) and (21) , we get

$$\sum_{n=2}^{\infty} \left[(n-1)(k+1)+1\right] \frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)} a_n$$

$$\leq \frac{(1-\gamma)(k+1)\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)} \sum_{n=2}^{\infty} \frac{(n-1)\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)}$$

$$+ \frac{(1-\gamma)\Gamma(\delta+1)\Gamma(2+\alpha)}{\Gamma(\beta+\delta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+\delta+n)\Gamma(n+1)}{(1-\lambda+n\lambda)\left[n(1+k)-(k+\gamma)\right]\Gamma(n+1+\alpha)\Gamma(\delta+n)} \qquad (22)$$

Finally , if we make use of the hypotheses (20) and (21) in (22) , we get

$$\sum_{n=2}^{\infty} \left[(n-1)(k+1)+1\right] \frac{\Gamma(2+\alpha)\Gamma(n+1)}{\Gamma(n+1+\alpha)} a_n \leq 1-\gamma \ .$$

So by using (11) and Lemma 4 , we get $f \in k-ST(\lambda,\gamma,\beta,\delta)$.
This complete the proof .

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Received: November, 2011