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On ((H – R)) Fractional Calculus

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Abstract

In the present paper , we have studied a class of univalent functions by applying a ((H – R)) fractional calculus , we obtain distortion theorem , wighted mean , arithmetic mean and some results .

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1. Introduction

Let R denote the class of functions of the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in IN = \{1,2,3,...\}) \qquad (1)$$

which are analytic and univalent in the unit disk

$$U = \{z \in \mathrm{C} : |z| < 1\}.$$

If $f \in$ R is given by (1) and $g \in$ R given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0 \ , \qquad (2)$$

then the Hadamard product (or convolution) $(f * g)$ of f and g is defined by

$$(f_* g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g_* f)(z) \qquad (3)$$

Lemma 1: [1]
The Rafid - Operator of $f \in$R for $0 \leq \mu < 1$, $0 \leq \theta \leq 1$ is denoted by R_μ^θ and defined as following :

$$R_\mu^\theta(f(z)) = \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)dt$$

$$= z - \sum_{n=2}^{\infty} K(n,\mu,\theta) a_n z^n, \qquad (4)$$

where $K(n,\mu,\theta) = \dfrac{(1-\mu)^{n-1}\Gamma(\theta+n)}{\Gamma(\theta+1)}$.

Lemma 2 :[3]
Let $\alpha \geq 0$, $0 \leq \beta < 1$ and $\gamma \in IR$. Then $\operatorname{Re} w > \alpha |w-1| + \beta$ if and only if $\operatorname{Re}(w(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}) > \beta$, where w be any complex number.

Lemma 3 :[3]
With the same condition as in Lemma 2 , $\operatorname{Re} w > \alpha$ if and only if
$|w-(1+\alpha)| < |w+(1-\alpha)|$.

Definition 1 : [1]
A function $f \in R$, $z \in U$ is said to be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ if and only if satisfies the inequality:

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f_* g)(z)))' + \lambda z^2 (R_\mu^\theta((f_* g)(z)))''}{(1-\lambda)R_\mu^\theta((f_* g)(z)) + \lambda z(R_\mu^\theta((f_* g)(z)))'}\right\} \geq$$
$$\beta\left|\frac{z(R_\mu^\theta((f_* g)(z)))' + \lambda z^2 (R_\mu^\theta((f_* g)(z)))''}{(1-\lambda)R_\mu^\theta((f_* g)(z)) + \lambda z(R_\mu^\theta((f_* g)(z)))'} - 1\right| + \alpha, \qquad (5)$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1, \beta \geq 0, z \in U, 0 \leq \mu < 1, 0 \leq \theta \leq 1$ and $g \in R$ given by (2).

Theorem 1: [1]
The function f defined by (1) is in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ if and only if

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)a_n b_n \le 1-\alpha, \qquad (6)$$

where $0\le\alpha<1, \beta\ge 0, 0\le\lambda\le 1, 0\le\mu<1$ and $0\le\theta\le 1$.

Lemma 4:
If $\operatorname{Re} w \ge \gamma|w-1|+k$, where $0\le k<1$, $\gamma\ge 0$. Then $|w|\ge\dfrac{\gamma+k}{\gamma+1}$.

Proof : Let $\operatorname{Re} w\ge\gamma|w-1|+k$, since $|w|\ge\operatorname{Re} w$, we get

$|w|\ge\gamma|w-1|+k$, or equivalent

$|w|(1+\gamma)\ge\gamma+k$, then $|w|\ge\dfrac{\gamma+k}{\gamma+1}$.

Definition 2 : [4]
Let $f(z)$ and $g(z)$ belong to R. Then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by

$$h_j(z)=\frac{1}{2}[(1-j)f(z)+(1+j)g(z)]. \qquad (7)$$

Definition 3 : [6]
The arithmetic mean of f_j $(j=1,2,...,q)$ is defined by

$$w(z)=\frac{1}{q}\sum_{j=1}^{q} f_j(z). \qquad (8)$$

In the next theorems we will show the weighted mean and arithmetic mean in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Theorem 2 :
If $f(z)$ and $g(z)$ are in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$. Then the weighted mean defined by Definition 2 is in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$, where

$$f(z)=z-\sum_{n=2}^{\infty} c_n z^n, \quad g(z)=z-\sum_{n=2}^{\infty} d_n z^n.$$

Proof : By Definition 2 ,we obtain

$$h_j(z)=\frac{1}{2}\left[(1-j)(z-\sum_{n=2}^{\infty} c_n z^n)+(1+j)(z-\sum_{n=2}^{\infty} d_n z^n)\right]$$

$$=z-\sum_{n=2}^{\infty}\frac{1}{2}[(1-j)c_n+(1+j)d_n]z^n.$$

We must show that $h_j(z)\in WR(\lambda,\beta,\alpha,\mu,\theta)$, so by Lemma 2 ,we get

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\alpha)\right]K(n,\mu,\theta)\left[\frac{1}{2}\left((1-j)c_n+(1+j)d_n\right)\right]b_n$$
$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\alpha)\right]K(n,\mu,\theta)\left[\frac{1}{2}(1-j)\right]c_n b_n$$
$$+\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\alpha)\right]K(n,\mu,\theta)\left[\frac{1}{2}(1+j)\right]d_n b_n$$
$$\leq \left[(1-j)+(1+j)\right](1-\alpha)=1-\alpha \ .$$

The proof is complete .

Theorem 3 : Let $f_j(z)$ defined by

$$f_j(z)=z-\sum_{n=2}^{\infty} a_{n,j}z^n \ \ (a_{n,j}\geq 0,\ j=1,2,\ldots,q) \tag{9}$$

be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$. Then the arithmetic defined by Definition 3 in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Proof : From (8) and (9) , we can write

$$w(z)=\frac{1}{q}\sum_{j=1}^{q} (z-\sum_{n=2}^{\infty} a_{n,j}z^n)$$
$$= z-\sum_{j=1}^{q} (\frac{1}{q}\sum_{n=2} a_{n,j})z^n \ .$$

Since $f_j(z)\in WR(\lambda,\beta,\alpha,\mu,\theta)$ for every $(j=1,2,\ldots,q)$, so by using Theorem 1, we get

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\alpha)\right]K(n,\mu,\theta)\left[\frac{1}{q}\sum_{n=2}^{q} a_{n,j}\right]b_n$$
$$=\frac{1}{q}\sum_{n=2}^{q}\left[\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\alpha)\right]K(n,\mu,\theta)a_{n,j}b_n\right]$$
$$\leq\frac{1}{q}\sum_{n=2}^{q} (1-\alpha)=(1-\alpha) \ .$$

The proof is complete .

Theorem 4 :

Let $f(z)$ defined by (1) be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$. Then

$$\left|\frac{z(R_{\mu}^{\theta}((f_{*}g)(z)))'+\lambda z^2(R_{\mu}^{\theta}((f_{*}g)(z)))''}{(1-\lambda)R_{\mu}^{\theta}((f_{*}g)(z))+\lambda z(R_{\mu}^{\theta}((f_{*}g)(z)))'}\right|\geq\frac{\beta+\alpha}{\beta+1} \ . \tag{10}$$

<u>Proof :</u> Since $f(z)\in WR(\lambda,\beta,\alpha,\mu,\theta)$, then by using Lemma 4 , we obtain

$$\left| \frac{z(R_\mu^\theta((f_* g)(z)))' + \lambda z^2 (R_\mu^\theta((f_* g)(z)))''}{(1-\lambda)R_\mu^\theta((f_* g)(z)) + \lambda z (R_\mu^\theta((f_* g)(z)))'} \right| \geq \frac{\beta+\alpha}{\beta+1} .$$

The proof is complete .

2.Application of a New Fractional Calculus

The fractional calculus connected with positive and negative coefficients. Such type of study was carried out by various mathematicians , like Aouf et. al. [2], Reddy and Padmanabhan [8], Atshan et. al.[6], Atshan and Kulkarni [4], who obtained several growth and distortion properties of functions in the class operators of fractional integral and fractional derivative.

Now , we introduce a new fractional calculus defined by ((H – R)) fractional calculus , ((fractional derivative and fractional integral)) of order δ .

Definition 4 :

The fractional integral of order δ $(\delta = 0,1,...)$ is defined by

$$ {}_z^{-\delta}Df(z) = \frac{1}{\Gamma(2(\delta+1))} \int_0^z (z-u)^{2\delta+1} f(u)du \ , \qquad (11)$$

where $f(z)$ is analytic function in simply connected region of $z-$plan containing the origin and the multiplicity of $(z-u)^{2\delta+1}$ is removed by required $\log(z-u)$ to be real when $(z-u) > 0$.

Definition 5 :

The fractional derivative of order δ $(\delta = 2,3,...)$ is defined by

$$ {}_z^{\delta}Df(z) = \frac{1}{\Gamma(\delta-1)} \int_0^z (z-u)^{\delta-2} f(u)du \ , \qquad (12)$$

where $f(z)$ is as Definition 4 and the multiplicity of $(z-u)^{\delta-2}$ is removed like Definition 4 .

From Definitions 4 and 5 applying a simple calculation, we get

$$\begin{aligned} G(z) &= z^{-2(\delta+1)}\Gamma(2+2(\delta+1))\, {}_z^{-\delta}Df(z) \\ &= z - \sum_{n=2}^{\infty} \frac{n!\,\Gamma(2+2(\delta+1))}{\Gamma(n+1+2(\delta+1))} a_n z^n . \qquad (13) \end{aligned}$$

And

$$F(z) = z^{2-\delta}\Gamma(\delta)\, {}_z^{\delta}Df(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n z^n . \qquad (14)$$

In the next , we obtain distortion theorems for the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Theorem 5: Let $f(z)$ defined by (1) be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$. Then

$$|G(z)| \le |z| + \frac{1-\alpha}{(2+\delta)(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2}|z|^2 \qquad (15)$$

and

$$|G(z)| \ge |z| - \frac{1-\alpha}{(2+\delta)(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2}|z|^2 \qquad (16)$$

Proof : By using Theorem 1, we get have

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\alpha}{(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2} . \qquad (17)$$

By Definition 4 , we have

$$G(z) = z - \sum_{n=2}^{\infty} \frac{n!\Gamma(2+2(\delta+1))}{\Gamma(n+1+2(\delta+1))} a_n z^n$$

$$= z - \sum_{n=2}^{\infty} \Phi(n,\delta) a_n z^n , \qquad (18)$$

where $\Phi(n,\delta) = \frac{n!\Gamma(2+2(\delta+1))}{\Gamma(n+1+2(\delta+1))}$.

We that $\Phi(n,\delta)$ is a decreasing function of n and $0 < \Phi(n,\delta) \le \Phi(2,\delta) = \frac{1}{2+\delta}$.

So by using (17) and (18), we have

$$|G(z)| \le |z| + \frac{1-\alpha}{(2+\delta)(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2}|z|^2$$

which gives (15), we also have

$$|G(z)| \ge |z| - \frac{1-\alpha}{(2+\delta)(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2}|z|^2$$

which gives (16) .

Theorem 6 : Let $f(z)$ defined by (1) be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$. Then

$$|F(z)| \le |z| + \frac{2(1-\alpha)}{\delta(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2}|z|^2 \quad \text{and}$$

$$|F(z)| \ge |z| - \frac{2(1-\alpha)}{\delta(1+\lambda)[2+\beta-\alpha](1-\mu)(\theta+1)b_2}|z|^2 .$$

The proof of theorem is similar to proof of theorem 5 .

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