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Some Geometric Properties of a Class of Univalent Functions with Negative Coefficients Defined by Hadamard Product with Fractional Calculus I

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Abstract

In this paper , we study a subclass of functions which are univalent and analytic functions in the unit disk U . We obtain coefficient estimates , distortion bounds and some results.

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1. Introduction

Let Ω denote the class of the functions defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ . \qquad (1)$$

Which are univalent and analytic in the unit disk $U=\{z\in C:|z|<1\}$. We defined a subclass K of Ω consisting of functions defined by

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \ . \quad (a_n\geq 0) \qquad (2)$$

A function $f(z)$ belong to the class $H(\alpha,\beta,\theta,\lambda)$ if satisfies

$$\operatorname{Re}\left\{\frac{z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''}{(1-\lambda)G(f_*g)(z))+\lambda z(G(f_*g)(z))'}\right\}\geq$$

$$\beta\left\{\frac{z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''}{(1-\lambda)G(f_*g)(z))+\lambda z(G(f_*g)(z))'}-1\right\}+\theta, \qquad (3)$$

where $0\leq\theta<1,\ 0\leq\lambda\leq 1,\beta\geq 0, z\in U, 0<\alpha<1$, and

$$g(z)=z-\sum_{n=2}^{\infty} b_n z^n \ , \ (b_n\geq 0) \qquad (4)$$

The Hadamard product or (convolution) defined by next definition .

Definition 1 :
If $f(z)$ defined by (2) and $g(z)$ defined by (4) . Then the Hadamard product or (convolution) defined by the form :

$$(f_*g)(z)=z-\sum_{n=2}^{\infty} a_n b_n z^n \ , \quad (a_n,b_n\geq 0) \qquad (5)$$

and $z\in U$.

Definition 2 : [3]
Fractional derivative of order α of analytic function $f(z)$ is defined by

$$D_z^{\alpha} f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\alpha}}dt \quad , \quad (0<\alpha<1) \qquad (6)$$

where $f(z)$ is an analytic function in a simply – connected region of the z – plane containing the origin, and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$ to be real , when $(z-t)$ is greater than zero . Clearly

$$f(z)=\lim_{\alpha\to 0} D_z^{\alpha} f(z)$$

and $$f'(z)=\lim_{\alpha\to 1} D_z^{\alpha} f(z) .$$

Definition 3: [6]
Fractional integral of order α of analytic function $f(z)$ is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt \quad , \tag{7}$$

where $f(z)$ is an analytic function in a simply – connected region of the $z-$plane containing the origin, and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real , when $(z-t)>0$ **.**

Definition 4: [6]
[Under the condition of Definition 3] the fractional derivative of order $n+\alpha$ $(n=0,1,2,...)$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^{\alpha} f(z), \tag{8}$$

For the analytic function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ in U we put

$$Mf(z) = \Gamma(2+\alpha) z^{-\alpha} D_z^{-\alpha} f(z)$$
$$= z - \sum_{n=2}^{\infty} \frac{n!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)} a_n z^n \quad , \quad \alpha > 0 \quad . \tag{9}$$

And

$$Gf(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z)$$
$$= z - \sum_{n=2}^{\infty} \frac{n!\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n \quad , \quad 0<\alpha<1 \quad . \tag{10}$$

Then , from (10) we get

$$G(f_* g)(z) = z - \sum_{n=2}^{\infty} \Psi(n,\alpha) a_n b_n z^n \quad , \tag{11}$$

where $$\Psi(n,\alpha) = \frac{n!\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \tag{12}$$

Lemma 2: [1]
Let $w = u + iv$. Then $\operatorname{Re} w \geq \sigma$ if and only if

$$|w-(1+\sigma)| \leq |w+(1-\sigma)|$$

Lemma 3: [1]
Let $w = u + iv$ and σ, γ are real numbers. Then
$\operatorname{Re} w > \sigma|w-1| + \gamma$ if and only if $\operatorname{Re}\{w(1+\sigma e^{i\phi}) - \sigma e^{i\phi}\} > \gamma$.

Some another class studied by W. G. Atshan and S. R. Kulkarni [2] ,S.Ponnusamy [4]H. M. Srivastava [5].

2. Coefficient Estimates

In the next theorem we get the sufficient condition for the function $f(z)$ in the class $H(\alpha,\beta,\theta,\lambda)$.

Theorem 1:
The function $f(z)$ defined by (2) is 'in the class $H(\alpha,\beta,\theta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\theta)]\Psi(n,\alpha)a_n b_n \le 1-\theta, \tag{13}$$

where $0\le\theta<1$, $0\le\lambda\le 1, \beta\ge 0, 0<\alpha<1$,

Proof:
By Definition 3 , we get

$$\operatorname{Re}\left\{\frac{z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''}{(1-\lambda)G(f_*g)(z))+\lambda z(G(f_*g)(z))'}\right\}\ge$$

$$\beta\left\{\frac{z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''}{(1-\lambda)G(f_*g)(z))+\lambda z(G(f_*g)(z))'}-1\right\}+\theta,$$

Then by Lemma 2 , we have

$$\operatorname{Re}\left\{\frac{z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''}{(1-\lambda)(G(f_*g)(z))+\lambda z(G(f_*g)(z))'}(1+\beta e^{i\gamma})-\beta e^{i\gamma}\right\}\ge\theta,$$
$$-\pi<\gamma\le\pi$$

or equivalently

$$\operatorname{Re}\left\{\frac{(z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''(1+\beta e^{i\gamma})}{(1-\lambda)(G(f_*g)(z))+\lambda z(G(f_*g)(z))'}\right.$$

$$\left.-\frac{\beta e^{i\gamma}\left[(1-\lambda)(G(f_*g)(z))+\lambda z^2(G(f_*g)(z))'\right]}{(1-\lambda)(G(f_*g)(z))+\lambda z(G(f_*g)(z))'}\right\}\ge\theta. \tag{14}$$

Let $A(z)=\left[(z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''(1+\beta e^{i\gamma})\right](1+\beta e^{i\gamma})$
$-\beta e^{i\gamma}\left[(1-\lambda)(G(f_*g)(z))+\lambda z^2(G(f_*g)(z))'\right]$

and $B(z)=\left[(1-\lambda)(G(f_*g)(z))+\lambda z(G(f_*g)(z))'\right]$.

By Lemma 1 , (14) is equivalent to

$$|A(z)+(1-\theta)B(z)|\ge|A(z)-(1+\theta)B(z)| \quad \text{for} \quad 0\le\theta<1$$

But $|A(z)+(1-\theta)B(z)|$

$$=\left|\left[z-\sum_{n=2}^{\infty}\Psi(n,\alpha)a_n b_n z^n-\lambda\sum_{n=2}^{\infty}n(n-1)\Psi(n,\alpha)a_n b_n z^n\right](1+\beta e^{i\gamma})\right.$$

$$-\beta e^{i\gamma}\left[(1-\lambda)(z-\sum_{n=2}^{\infty}\Psi(n,\alpha)a_nb_nz^n)+\lambda z-\lambda\sum_{n=2}^{\infty}n\Psi(n,\alpha)a_nb_nz^n\right]$$

$$+(1-\theta)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Psi(n,\alpha)a_nb_nz^n\right]\Bigg|$$

$$=\Bigg|(2-\theta)z-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1))+(1-\theta)(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_nz^n$$

$$-\beta e^{i\gamma}\sum_{n=2}^{\infty}\left[n+\lambda n(n-1)-(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_nz^n\Bigg|$$

$$\geq(2-\theta)|z|-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1))+(1-\theta)(1-\lambda+\lambda n)\right]\Psi(n,\alpha)a_nb_n|z|^n$$

$$-\beta\sum_{n=2}^{\infty}\left[n+\lambda n(n-2)-1+\lambda\right]\Psi(n,\alpha)a_nb_n|z|^n$$

Also $|A(z)-(1+\theta)B(z)|=\Bigg|\left[z-\sum_{n=2}^{\infty}n\Psi(n,\alpha)a_nb_nz^n\right.$

$$\left.-\lambda\sum_{n=2}^{\infty}n(n-1)\Psi(n,\alpha)a_nb_nz^n\right](1+\beta e^{i\gamma})$$

$$-\beta e^{i\gamma}\left[z-(1-\lambda)\sum_{n=2}^{\infty}\Psi(n,\alpha)a_nb_nz^n-\lambda\sum_{n=2}^{\infty}n\Psi(n,\alpha)a_nb_nz^n\right]$$

$$-(1+\theta)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Psi(n,\alpha)a_nb_nz^n\right]\Bigg|$$

$$=\Bigg|-\theta z-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1))-(1+\theta)(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_nz^n$$

$$-\beta e^{i\gamma}\sum_{n=2}^{\infty}\left[n+n\lambda(n-1)-(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_nz^n\Bigg|$$

$$\leq\theta|z|+\sum_{n=2}^{\infty}\left[(n+n\lambda(n-1))-(1+\alpha)(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_n|z|^n$$

$$+\beta\sum_{n=2}^{\infty}\left[n+n\lambda(n-1)-(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_n|z|^n$$

and so $|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\geq 2(1-\theta)|z|$

$$-\sum_{n=2}^{\infty}\left[(2n+2n\lambda(n-1))-2\theta(1-\lambda+n\lambda)-\beta(2n+2n\lambda(n-1)-\right.$$

$$\left.-2(1-\lambda+n\lambda))\right]\Psi(n,\alpha)a_nb_n|z|^n\geq 0$$

or

$$\sum_{n=2}^{\infty}\left[n(1+\beta)+n\lambda(n-1)(1+\beta)-(1-\lambda+n\lambda)(\theta+\beta)\right]\Psi(n,\alpha)a_nb_n\le 1-\alpha.$$

This is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)a_nb_n\le 1-\theta.$$

Conversely , suppose that (2.1) holds. Then we must show

$$\mathrm{Re}\left\{\frac{(z(G(f_*g)(z))'+\lambda z^2(G(f_*g)(z))''(1+\beta e^{i\gamma})}{(1-\lambda)(G(f_*g)(z))+\lambda z(G(f_*g)(z))'}\right.$$

$$\left.-\frac{(\theta+\beta e^{i\gamma})\left[(1-\lambda)(G(f_*g)(z))+\lambda z^2(G(f_*g)(z))'\right]}{(1-\lambda)(G(f_*g)(z))+\lambda z(G(f_*g)(z))'}\right\}\ge 0.$$

Upon choosing the values of z on the positive real axis where $0\le z=r<1$, the above inequality reduces to

$$\mathrm{Re}\left\{\frac{(1-\theta)-\sum_{n=2}^{\infty}\left[n(1+\beta e^{i\gamma})(1-\lambda+\lambda n)-(\theta+\beta e^{i\phi})(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Psi(n,\alpha)a_nb_nr^{n-1}}\right\}\ge 0.$$

Since $\mathrm{Re}\left(-e^{i\gamma}\right)\ge-\left|e^{i\gamma}\right|=-1,$ the above inequality reduces to

$$\mathrm{Re}\left\{\frac{(1-\theta)-\sum_{n=2}^{\infty}\left[n(1+\beta)(1-\lambda+\lambda n)-(\theta+\beta)(1-\lambda+n\lambda)\right]\Psi(n,\alpha)a_nb_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Psi(n,\alpha)a_nb_nr^{n-1}}\right\}\ge 0.$$

Letting $r\to 1^-$, we get desired conclusion .

Corollary 1 :

Let $f(z)\in H(\alpha,\beta,\theta,\lambda).$ Then $a_n\le\dfrac{1-\theta}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))\Psi(n,\alpha)b_n}$,

3. Distortion Theorem

In the next theorem , we obtain the distortion theorem for $f(z)\in H(\alpha,\beta,\theta,\lambda).$

Theorem 2:

If $f(z)\in H(\alpha,\beta,\theta,\lambda).$ Then

$$|z|-|z|^2\frac{(1-\theta)(3-\alpha)}{2(1+\lambda)\left[2(1+\beta)-(\beta+\theta)b_n\right]}\le|f(z)|\le|z|+|z|^2\frac{(1-\theta)(3-\alpha)}{2(1+\lambda)\left[2(1+\beta)-(\beta+\theta)b_n\right]}$$

Proof :

Since $|f(z)|\le|z|+|z|^2\sum_{n=2}^{\infty}a_n$,

from (13) , we get

$$\sum_{n=2}^{\infty}a_n\le\frac{(1-\theta)(3-\alpha)}{2(1+\lambda)\left[2(1+\beta)-(\beta+\theta)b_n\right]}\quad, \tag{15}$$

hence

$$|f(z)|\le|z|+|z|^2\frac{(1-\theta)(3-\alpha)}{2(1+\lambda)\left[2(1+\beta)-(\beta+\theta)b_n\right]}.$$

Similarity , we get

$$|f(z)|\ge|z|-|z|^2\frac{(1-\theta)(3-\alpha)}{2(1+\lambda)\left[2(1+\beta)-(\beta+\theta)b_n\right]}.$$

In the next theorem , we shall prove that the class $H(\alpha,\beta,\theta,\lambda)$ is closed under arithmetic mean and convex linear combinations .

Now , we defined the function $f_k(z)$ by the form

$$f_k(z)=z-\sum_{n=2}^{\infty}a_{n,k}z^n\ ,\ (a_{n,k}\ge0,n\in IN) \tag{16}$$

Theorem 3:

Let the function $f_k(z)$ defined by (16) be in the class $H(\alpha,\beta,\theta,\lambda)$ for $(k=1,2,...,m)$,. Then the function

$$\Phi(z)=z-\sum_{n=2}^{\infty}c_nz^n\ ,\ (c_n\ge0,n\in IN) \tag{17}$$

Also in the class $H(\alpha,\beta,\theta,\lambda)$, where $c_n=\frac{1}{m}\sum_{k=1}^{m}a_{n,k}$.

Proof :

Let the function $f_k(z)\in H(\alpha,\beta,\theta,\lambda)$, then from theorem 1 , we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)a_{n,k}b_n\le1-\theta\quad.$$

Hence

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)c_nb_n$$

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)b_n\left[\frac{1}{m}\sum_{k=1}^{m} a_{n,k}\right]\le 1-\theta \quad .$$

Hence $\Phi(z)\in H(\alpha,\beta,\theta,\lambda)$.

Theorem 4:
The class $H(\alpha,\beta,\theta,\lambda)$ is closed under linear combinations .

Proof :
Let the function $f_k(z)$ $(k=1,2)$ defined by (16) be in the class $H(\alpha,\beta,\theta,\lambda)$. Now we show that the next function $E(z)=\ell f_1(z)+(1-\ell)f_2(z)$, $(0\le \ell \le 1)$ is also in the class $H(\alpha,\beta,\theta,\lambda)$. Since $f_1(z)\in H(\alpha,\beta,\theta,\lambda)$ then from (13) , we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)a_{n,1}b_n \le 1-\theta .$$

And so $f_2(z)\in H(\alpha,\beta,\theta,\lambda)$ then from (13) we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)a_{n,2}b_n \le 1-\theta \quad .$$

Then

$$E(z)=z-\sum_{n=2}^{\infty}\left[\ell a_{n,1}+(1-\ell)a_{n,2}\right]z^n \quad .$$

Therefore by Theorem 1, we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\theta)\right]\Psi(n,\alpha)b_n\left[\ell a_{n,1}+(1-\ell)a_{n,2}\right]\le 1-\theta .$$

Hence by Theorem 1, we have $E(z)\in H(\alpha,\beta,\theta,\lambda)$.

Theorem 5: Let the function $f_k(z)$ defined by (16) be in the class $H(\alpha,\beta_k,\theta_k,\lambda_k)$ where $(0\le\theta_k\le 1,\beta_k\ge 0, 0<\alpha<1, 0\le\lambda\le 1, n\ge 2)$. For each $(k=1,2,...,m)$, then the function

$$S(z)=z-\frac{1}{m}\sum_{n=2}^{\infty}\left[\sum_{k=1}^{m} a_{n,k}\right]z^n$$

is also in the class $H(\alpha,\beta,\theta,\lambda)$, where
$\beta=\min_{1\le k\le m}\{\beta_k\}$, $\theta=\min_{1\le k\le m}\{\theta_k\}$ and $\lambda=\min_{1\le k\le m}\{\lambda_k\}$.

Proof : Let the functions $f_k(z) \in H(\alpha, \beta_k, \theta_k, \lambda_k)$, then from Theorem 1 we get

$$\sum_{n=2}^{\infty}(1-\lambda_k + n\lambda_k)\left[n(1+\beta_k)-(\beta_k+\theta_k)\right]\Psi(n,\alpha)a_{n,k}b_n \le 1-\theta_k ,$$

hence

$$\sum_{n=2}^{\infty}(1-\lambda_k + n\lambda_k)\left[n(1+\beta_k)-(\beta_k+\theta_k)\right]\Psi(n,\alpha)b_n\left[\frac{1}{m}\sum_{k=1}^{m} a_{n,k}\right] \le \frac{1}{m}\sum_{k=1}^{m}(1-\theta_k)$$
$$\le 1-\theta .$$

Therefore $S(z) \in H(\alpha, \beta, \theta, \lambda)$.

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