INTERNATIONAL JOURNAL OF ADVANCED RESEARCH IN ENGINEERING AND TECHNOLOGY (IJARET)

ISSN 0976 - 6480 (Print)
ISSN 0976 - 6499 (Online)
Volume 5, Issue 4, April (2014), pp. 141-145
© IAEME: www.iaeme.com/ijaret.asp
Journal Impact Factor (2014): 7.8273 (Calculated by GISI)
www.jifactor.com

IJARET

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ON THE SEMI TRANSITIVE MAPS

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ABSTRACT

Let (X, f) be dynamical system, (X, f) is semi transitive if for every pair of non-empty semi open subset $J.K$ in X, there is a positive integer n such that $f^n(J) \cap K \neq \phi$. In this works we study semi transitive map and generalize some equivalent definitions of semi transitive.

1- INTRODUCTION

Let X be a metric space and let the distance between two points $x, y \in X$ be denoted $d(x, y)$. A discrete system at its simplest is the set of iterates of a map $f : X \to X$, i.e.,$\{f^0, f^1, f^2, ...\}$, where f^0 is the identity function and f^n denotes f composed with itself n times. The orbit of a point $x \in X$ is the set $\{x, f(x), f^2(x), ...\}$ and will often be written $O_f(x)$. A point x is said to be periodic if $f^n(x) = x$ for some positive n. The minimum such n is called the period of x.

A discrete dynamical system is a pair (X, f), where X is a topological space and f is continuous self map on X. If for every pair of non-empty open sets U & V in X, there is a $n \in N$ such that $f^n(U) \cap V \neq \phi$, then the system (X, f) is said to be topologically transitive[2]. Many times, the system is said to be transitive if there is an $x \in X$ such that $\overline{O_f(x)} = X$ (i.e., X has a dense orbit)[4]. Both, these definition of transitivity are equivalent, in wide class of spaces, including all connected compact metric spaces[2].

A dynamical system is said to be semi transitive, If for every pair of nonempty semi-open K & J, there is a $n \in N$ such that $f^n(K) \cap J \neq \phi$[3]. It easy to show that every transitive is semi transitive but the converse is not true.

Recall that we denote the set of all periodic points of f by $p(f)$. If $p(f)$ is dense in X, we sometimes say that f or (X,f) has dense periodicity. Of course, dense periodicity cannot imply the transitivity if the phase space has more than one point. But there space where transitivity implies dense periodicity [5].

Sensitivity is yet another dynamical property. It signifies instability of the system. Recall its definition: let (X,f) be a metric space, the $f:X \to X$ is said to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in X$ and any neighborhood N_x of x, $\exists y, y \in N_x$ and a $n \in N$ such that $d(f^n(x), f^n(y)) > \delta$. The relation with transitive became clear after a popular definition of chaos [4] which put them both, a long with a third condition, denseness of periodic orbits, as defining characteristic of "chaos". This relation became more prominent when it was deduced that transitivity along with the denseness of periodic orbits actually implied sensitivity, giving a redundancy in the definition

Theorem(1-1)[4]:

Let X be an infinite metric space and $f:X \to X$ be continuous. If f is transitive and has a dense set of periodic points then f has sensitive dependence on initial conditions.

Silverman [4], shows some result about transitive, in this work, we generalize some these

Theorem(1-2)[5]:

Let X be a perfect, then dense orbit implies topological transitive. Furthermore, if X is separable and second category, then topological transitive implies dense orbit.

Theorem(1-3)[5]:

If X is infinite, then dense orbit and dense periodic point imply sensitive dependent on initial condition.

2. SEMI TRANSITIVE

Let (X,f) be a dynamical system, f is said to be semi transitive if , for every pair of semi open set K,J , $f^n(K) \cap J \neq \phi$,for all $n \in N$. It easy to show that every transitive map is semi transitive but the converse is not true[3]. Now we prove the following result.

Theorem(2-1):

Let (X,f) be a dynamical system, then the following are equivalent:

1- f is topological semi transitive.

2- For every nonempty semi open set J in X, $\bigcup_{n=0}^{\infty} f^n(J)$ is semi dense in X.

3- For every nonempty semi open set J in X, $\bigcup_{n=0}^{\infty} f^{-n}(J)$ is semi dense in X.

Proof:

$1 \Rightarrow 2$. Assume $\bigcup_{n=0}^{\infty} f^n(J)$ is not semi-dense. Then there exist a nonempty semi-open K such that $f^n(J) \cap K \neq \phi$. This implies $f^n(J) \cap K \neq \phi$ for all $n \in N$. This a contradiction to the semi-transitivity of f.

2 ⇒ 1 . Let J and K be two nonempty semi-open sets in X.$\bigcup_{n=0}^{\infty} f^{n}(J)$ is semi-dense in X . $\Rightarrow \bigcup_{n=0}^{\infty} f^{n}(J) \cap K \neq \phi$ .This implies there exist $m \in \mathbb{N}$ such that $f^{m}(J) \cap K \neq \phi$ .Hence f is semi transitive.

1 ⇒ 3 $\bigcup_{n=0}^{\infty} f^{-n}$ (J) is semi-open and since f is semi transitive, it has to meet every semi-open set in X and hence is semi-dense.

3 ⇒ 1 Let J and K be two nonempty semi-open sets in X.$\bigcup_{n=0}^{\infty} f^{-n}(J)$ is semi-dense in X. As a result $J \cap f^{-n}(K) \neq \phi$. This implies $\exists m \in N$ such that $J \cap f^{-n}(K) \neq \phi$. We further have $f^{m}(J \cap f^{-n}(K)) = f^{m}(J) \cap K \neq \phi$. Hence f is semi-transitive.
Now, we can prove the theorem(1-2) for semi transitive map

Theorem(2-2):

let X be a perfect ,then semi dense orbit implies semi topological transitive. Furthermore, if X is separable and second category, then topological semi transitive implies semi dense orbit.

Proof:

Part one: let (X, f) be dynamical system such that X has no isolated point, and has a dense orbit. Let $x_{\circ} \in X$ such that $O_f(x_{\circ})$ is semi dense in X . Let J be a non-empty semi open set in X , then $\exists k \in \mathbb{N}$ such that $f^{k}(x_{\circ}) \in J$ .Let K be anther non-empty semi open set in X , consider $W = K \setminus \{x_{\circ}, x_1, ..., x_k\}$, where $x_j = f^{j}(x_{\circ})$. Clearly W is non-empty and semi open. Then $\exists m \in \mathbb{N}$ and $m > k$ such that $f^{m}(x_{\circ}) \in W$. This implies for $n = m - k, f^{n}(J) \bigcap K \neq \phi$. This proves that f is semi transitive.

Part two: Let (X, f) be a dynamical system, X is separable, second category and f is semi transitive. Assume that there is no semi dense orbit. Let $\{k_n\}_{n=1}^{\infty}$ be a countable base for X . This implies, $\forall x \in X \;\; \exists J_x$, non-empty semi open set such that $O_f(x) \bigcap J_x = \phi$, which implies $\exists K_{n(x)} \subset J_x$ such that $O_f(x) \bigcap K_{n(x)} = \phi$. Further, by Theorem(2-1),we have $\bigcup_{n=0}^{\infty} f^{-k}(K_{n(x)})$ is semi dense in X .Because $\bigcup_{n=0}^{\infty} f^{-k}(K_{n(x)})$ is semi open and semi transitive demands that is should meet every semi open set in X .

Define $A_{n(x)} = X \setminus \bigcup_{k=0}^{\infty} f^{-k}(K_{n(x)})$ .Clearly $A_{n(x)}$ is semi closed, nowhere dense and $x \in A_{n(x)}$. Thus $X = \bigcup_{n(x)=1}^{\infty} A_{n(x)}$, a countable union of nowhere dense sets. This is a contradiction to the fact that X is a second category space. Hence (X, f) has a semi dense orbit.

The study of chaotic dynamical systems has become increasingly popular nowadays Although there has been no universally accepted mathematical definition of chaos, it is generally believed that sensitive dependence on initial conditions is the central element of chaos (see also [1]).

Therefore, it would be interesting to know under what conditions, sensitive dependence on initial conditions can be guaranteed when the system is semi transitive.

Theorem(2-3):

If X is infinite, then semi dense orbit and semi dense periodic point imply sensitive dependent on initial condition.

Proof:

Since $P(f) \neq \phi$, we have $P(f)$ is semi dense in X.Let us choose two periodic points q_1, q_2 such that $O_f(q_1) \cap O_f(q_2) = \phi$. Let $\delta_\circ = d(O_f(q_1), O_f(q_2))$, set $\delta = \frac{\delta_\circ}{8}$.

Notice that $\delta_\circ > o$, and for every $x \in X$, other $d(x, O_f(q_1)) > \frac{\delta_\circ}{2}$ or $d(x, O_f(q_2)) > \frac{\delta_\circ}{2}$.

Let $x \in X$ and J be any semi open set that includes x. Let $N_\delta(x)$ be a neighborhood of x with radius δ.Let p be a periodic point in $W = J \cap N_\delta(x)$ with period n. From this we conclude that one of the points q_1, q_2 (denoted q) has an orbit ,for which $d(x, O_f(q)) > 4\delta$.

Let us define $V = \bigcap_{i=0}^{n} f^{-i}(N_\delta(f^i(q)))$ .The set V is non-empty, because $q \in V$ and V is open from the semi transitivity of f there exists a point $y \in W$ and integer number k such $f^k(y) \in V$. Let j be integer part of $\frac{k}{n} + 1$. Consequently, $\frac{k}{n} + 1 = j + r$, when r is the rest, $o \leq r < 1$. Clearly, $nj - k = n - rn$.It follows that $0 < nj - k \leq n$.

By construction, $f^{nj}(y) = f^{nj-k}(f^k) \in f^{nj-k}(V) \subset N_\delta(f^{nj-k}(q))$

Let $a = f^{nj}(y)$

$b = f^{nj-k}(q)$

Note that, $d(a,b) < \delta$, let us use the triangle inequality for points p, a, b and x, p, b,

$d(p,b) \leq d(p,a) + d(a,b)$

$d(x,b) \leq d(x,p) + d(p,b)$

when, $d(x,b) \leq d(x,p) + d(p,a) + d(a,b)$

or $d(p,a) \geq d(x,b) - d(x,p) - d(a,b)$

By construction,

$d(x,b) = d(x, f^{nj-k}(q)) \geq d(x, O_f(q)) \geq 4\delta$.

Since $p \in N_\delta(x)$, then $d(x,p) < \delta$. From this it follows that $d(p,a) > 4\delta - \delta - \delta$, or $d(f^{nj}(p), f^{nj}(y) > 2\delta$.

Applying the triangle inequality to the following points $f^{nj}(x), f^{nj}(p), f^{nj}(y)$, we get that: $d(f^{nj}(x), f^{nj}(p)) > \delta$ or $d(f^{nj}(x), f^{nj}(y)) > \delta$

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